Series of Chromatic Differences

Gilbert G. Walter
UW-Milwaukee
February 2014
Outline of talk

Taylor's Series—what's wrong?
History of chromatic derivatives and series
What's wrong with them?
Extension to Slowly growing BL signals
Chromatic Differences and Series
(i) $f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)t^n}{n!}$ converges only locally.

(ii) Representation of bandlimited functions not bandlimited

Ignjatovic (1990) used other derivatives (chromatic derivatives) $p_n(-iD)f(0)$, not $f^{(n)}(0)$

${\{p_n(x)\}}$ orthogonal polynomials wrt weight $w(x)$
Chromatic series

Taylor series replaced by series

\[ f(t) = \sum_{n=0}^{\infty} 2\pi \{p_n(-iD)f\}(0) \varphi_n(t), \]

\( \varphi_n(t) \) inverse Fourier transform

\[ \varphi_n(t):=(1/(2\pi)) \int_{-\infty}^{\infty} e^{i\omega t} p_n(\omega) w(\omega) d\omega \]

convergence uniform on all of \( \mathbb{R} \).

(provided \( w \) has c.s.)
\( \varphi_n(t) \) takes the place of \( t^n/n! \) in Taylor series

\( \varphi_0, \varphi_3, \varphi_6 \) look like this:
Here’s how CS works:

Let \( g(t) \) be function in \( B_{\pi} \), \( w \) supported in \([-\pi, \pi]\)

Take polynomial expansion of F.T. \( \hat{g}(\omega) \) in form

\[
\hat{g} = \sum_n \left\{ \int \hat{g}(\omega) p_n(\omega) \, d\omega \right\} p_n \, w
\]

Take inverse Fourier transform

\[
g(t) = \sum_n \int_{-\pi}^{\pi} \hat{g}(\omega) p_n(\omega) \, d\omega \varphi_n(t)
\]

But

\[
\int_{-\pi}^{\pi} \hat{g}(\omega) p_n(\omega) \, d\omega = 2\pi \{p_n(-iD)g\}(0)
\]

since

\[
\int_{-\pi}^{\pi} \hat{g}(\omega) \omega^n \, d\omega = \int_{-\pi}^{\pi} e^{i\omega t} \hat{g}(\omega) \omega^n \, d\omega \bigg|_{t=0} = 2\pi \{(-iD)^n g\}(0)
\]
Chromatic series

are globally convergent (for \( f \) bandlimited)

are bandlimited (if \( w \) has compact support)

in contrast to Taylor series
Example: Legendre Polynomials

\[ w(\omega) = \chi_{[-1,1]}(\omega), \]

\[ P_0(\omega) = 1, \ P_1(\omega) = \omega, \ldots, \]

\[ (n+1)P_{n+1}(\omega) = (2n+1)\omega P_n(\omega) - nP_{n-1}(\omega), \]

\[ \varphi_n(t) := \frac{1}{(2\pi)} \int_{-1}^{1} e^{i\omega t} P_n(\omega) d\omega \sqrt{\|P_n\|^2} \]

(Spherical Bessel Function)
What's Wrong?

Need input \( \{p_n(-iD)g\}(0) \)

Need to compute
\[
\varphi_n(t) := \frac{1}{(2\pi)} \int_{-\infty}^{\infty} e^{i\omega t} p_n(\omega) w(\omega) d\omega
\]

Paley - Wiener space \( B_{\pi} \) doesn't include all signals; e.g., periodic signals, polynomials.
Extending Paley-Wiener Space

Denote by \( B_{\pi}^{-m} \), \( m \) integer \( \geq 0 \),

\[ \{ g \in C(\mathbb{R}) / \hat{g} \in S' \text{ of order } m \text{ with support in } [-\pi, \pi] \}. \]

\( B_{\pi}^{-m} \) includes periodic signals, polynomials, for \( m > 1 \).
Example 1

Let \( f(t) = t^j \) for some positive integer \( j \).

Fourier transform of \( t^j \) is \( 2\pi i^j \delta^{(j)} \)

and has support \( \{0\} \subset [-\pi, \pi] \)
Chromatic Derivatives in $B_{\pi}^{-m}$

Computations the same in $B_{\pi}^{-m}$

i.e., $\{p_n(-iD)g\}(0), \varphi_n(t)$ still needed.

Convergence weaker;

in sense of $S'$ (tempered distributions).
Thm. Let \( f \in B^{−m}_\pi, m \) integer \( \geq 0; \)

then chromatic series of \( f \) converges

in sense of \( S' \) to \( f \).

Not very useful, better to get some pointwise convergence
Uniform convergence

**Thm.** Let $f(z)$ be given by a convergent power series for $|z| < r$;

then $f(z)$ has chromatic series uniformly convergent to $f(z)$ on compact subsets of disk.
Examples

\[ f_2(t) = \sin(t/2), \text{ then } f_2 \in B^{-1}_{\pi}, \]

\[ f_4(t) = t^3, \text{ then } f_4 \in B^{-4}_{\pi}. \]
$f_2$ with 12 term partial sum of c.s.
$f_4(t) = t^3$, 3 and 4 term c.s.
Different approach: Chromatic Differences; polynomials orthogonal on circle

Start with $1, z, z^2, \ldots$ and orthogonalize on $\{|z|=1\}$ with respect to weight function $\nu(z)/z$,
where $\nu(e^{i\theta})\chi_{\pi}(\theta)=w(\theta) \geq 0$ on $[-\pi, \pi]$

Denote by $\{p_n(z)\}$ resulting orthogonal system.
Let $p_n(z) = \sum_{k=0}^{n} c_k^n z^n$; let $h(t)$ be $\pi$ bandlimited;

Then $a_n = \sum_{k=0}^{n} c_k^n (h^{*}w^{-1})(k)$ are the Chromatic Differences
Let $\psi_n(t) := \frac{1}{2\pi} \int e^{i\omega t} p_n(e^{-i\omega t}) w(\omega) d\omega$.

Then $\sum_{n=0}^{\infty} a_n \psi_n(t)$ is

**Discrete chromatic series**

of $h(t)$. 
Example

Take \( w(\theta) = \chi_\pi(\theta) \)

then \( p_n(z) = z^n \), or \( p_n(e^{i\theta}) = e^{in\theta}, n = 0, 1, \ldots \).

and \( h(t) = \sum_{n=0}^{\infty} h(n)s(t-n), \)

where \( s(t) \) is sinc function.
Problem:

Discrete CS of $h(t)$ converges in sense of Paley-Wiener space $B_\pi$, but doesn't always converge to $h(t)$. 
Decomposition of $B_{\pi}$

Note example includes only non-negative terms of exponential trig functions.

Define $B_{\pi}^+ = \{ f \in B_{\pi} | \hat{f} \in H^2[-\pi, \pi] \}$ for $w > 0$ on $[-\pi, \pi]$.

For general $w$, define $B_w^+ = \{ f \in B_{\pi} | f/w \in H^2[-\pi, \pi] \}$.
Prop. Let $h \in B_w^+$, $g = \hat{h}/w$, $p_n(z) = \sum_{k=0}^{n} c_k^n z^n$,

$\tilde{g}$ be inverse FT of $g$, $\psi_n$ inv. FT of $p_n(e^{-i\omega t}) w$,

$a_n = \sum_{k=0}^{n} c_k^n \tilde{g}(k)$;

then $\sum_{n=0}^{\infty} a_n \psi_n(t)$ converges to $h(t)$

uniformly on compact subsets of $\mathbb{R}$. 
More examples:

1) \( w(\theta) = ((1 + \cos \theta)/2) \chi_{\pi}(\theta) \), \quad \text{(Raised cosine)}

Then inv. FT is \( \psi(t) = (\sin \pi t)/2\pi t(1-t^2) \) and
\[
\psi_n(t) = \sum_{k=0}^{n} c_k^n \psi(t-k)
\]

2) \( w(\theta) = (1 - \cos^2 \theta)\lambda \chi_{\pi}(\theta), \quad \lambda > 0 \)
(leads to Gegenbauer polynomial based \( p_n(z) \))

**Problem:** Result only holds for \( B^+_w \) not for all of \( B^+_{\pi} \)
Symmetric weight:

Then \( \{p_n(z^{-1})| n=1,2,..\} \) is also orthogonal system on circle.

Combine two systems by setting
\[
p_{-n}(z) = p_n(z^{-1}), \quad n=1,2,...
\]
to get system \( \{p_n(z)| n=0, \pm 1, \pm 2,..\} \)

Then \( \{p_n(e^{i\theta})\}_{n=-\infty}^{\infty} \) is Riesz basis of \( L^2(w,[-\pi,\pi]) \) and \( \{\psi_n(t)\} \) is Riesz basis of \( B_{\pi} \)

(under certain conditions on \( w \))
Other approach:

Orthogonalize $1, z^1, z^{-1}, z^2, \ldots$ on unit circle with respect to weight $w(\theta)=\alpha(e^{i\theta})$ to get orthonormal system $\{\varphi_n\}$.

Prop. Let $w(\theta)=w(-\theta) \geq 0$ on $[-\pi, \pi]$, then $\{\varphi_n\}$ is orthonormal basis of $L^2(w, [-\pi, \pi])$ and

$$\varphi_n(e^{i\theta}) = \sum_{k=-|n|}^{|n|} a_{k,n} e^{ik\theta}.$$
Discrete chromatic series on $B_\pi$

**Thm.** Let $h \in B_\pi$ with $g=\hat{h}/w \in L^2 [-\pi,\pi]$; then

$$h(t) = \sum_{n=-\infty}^{\infty} \sum_{k=-|n|}^{|n|} a_{k,n} g(k) \psi_k(t)$$

where

$$\psi_k(t) = \frac{1}{2\pi}\int e^{i\omega t} \varphi_k(e^{-i\omega t}) w(\omega) d\omega$$

and convergence is in $L^2(\mathbb{R})$ and uniformly in $\mathbb{R}$. 
Thm. Let $f \in B_{\pi-\epsilon}^{-m}$, $m$ integer $\geq 0$, let $w$ be trig polynomial
\[ w(\theta) > 0 \text{ on } (-\pi, \pi) \quad \& \quad w^{(k)}(\pm \pi) = 0, k \leq m; \]

Then discrete chromatic series of $f$ converges in sense of $S'$ to $f$ and uniformly on compact sets.
Some references


