Shift Invariant Spaces and BMO

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Fourier Talks, University of Maryland, February 20-21, 2014
Given a function $\psi \in L_2(\mathbb{R}^d)$, one of the most basic operations we can consider is translation: $T_k \psi := \psi(\cdot - k)$, $k \in \mathbb{Z}^d$.

Translation is a fundamental operator in harmonic analysis since it is ”simple” and behaves well under the Fourier transform:

$$\mathcal{F}(T_k \psi) = e^{-2\pi ik \cdot \hat{\psi}}, \quad \text{where } \mathcal{F}(f)(\xi) := \int_{\mathbb{R}} f(x)e^{-2\pi ix \cdot \xi} \, dx.$$
A finitely generated shift-invariant (FSI) subspaces of $L_2(\mathbb{R}^d)$ is a subspace $S \subset L_2(\mathbb{R}^d)$ for which there exists a finite family $\Psi$ of $L_2(\mathbb{R}^d)$-functions such that

$$S = S(\Psi) := \text{span}\{\psi(\cdot - k) : \psi \in \Psi, k \in \mathbb{Z}^d\}.$$ 

**Remark**

To keep the notation simple, we only consider the most basic case: $d = 1$ and $\#\Psi = 1$ [PSI space].

**Applications**

FSI/PSI subspaces are used in several applications.

- Wavelets and other multi-scale methods are based on PSI subspaces
- FSI/PSI subspaces play an important role in multivariate approximation theory such as spline approximation and approximation with radial basis functions.
Stable generating set

Given the structure of \( S \), it is natural to consider a generating sets of integer translates. That is, a system with the following structure,

\[
\{ \varphi(\cdot - k) : k \in \mathbb{Z} \},
\]

Often we take \( \varphi = \psi \), but \( \varphi \) may be different from \( \psi \). However, we always require that \( S(\varphi) = S(\psi) \).
Basic Fourier Analysis of $S(\psi)$

It can easily be deduced from the identity,

$$f = \sum_k c_k \psi(\cdot - k) \Rightarrow \hat{f} = \sum_k c_k e^{-2\pi i k \cdot \hat{\psi}}$$

$$\Rightarrow \|\hat{f}\|_2^2 = \int_\mathbb{T} \left| \sum_k c_k e^{-2\pi i k \xi} \right|^2 \sum_j |\hat{\psi}(\xi + j)|^2 \, d\xi$$

that

$$J_\psi m := (m \cdot \hat{\psi})^\vee$$

is an isometry from $L_2(\mathbb{T}; p_\psi)$ onto $S(\psi)$, where $p_\psi$ is the periodization of $|\hat{\psi}|^2$, given by

$$p_\psi(\xi) := \sum_{k \in \mathbb{Z}} |\hat{\psi}(\xi + k)|^2, \quad \xi \in \mathbb{R}.$$ 

Observation

The system $\{e^{2\pi i k \xi}\}_k$ in $L_2(\mathbb{T}; p_\psi)$ is mapped by $J_\psi$ to $\{\psi(\cdot - k)\}_k$. 

M. Nielsen

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Some well-known classical results

**Orthonormal and Riesz bases**

Let $\psi \in L_2(\mathbb{R})$ and consider

$$B := \{ \psi(\cdot - k) : k \in \mathbb{Z} \}.$$ 

We let

$$p_\psi(\xi) := \sum_{k \in \mathbb{Z}} |\hat{\psi}(\xi + k)|^2, \quad \xi \in \mathbb{R}.$$ 

Then

- $B$ forms an orthonormal basis for $S(\psi)$ provided $p_\psi \equiv 1$.
- $B$ forms a Riesz basis for $S(\psi)$ provided that $p_\psi \asymp 1$.

**Extension to FSI spaces**
The above result can be extended to FSI spaces using the Grammian for the generating set.

**Question**

Is stability of $B$ possible even if $p_\psi \not\asymp 1$?
Weaker notion of stability: Schauder bases

**Definition**

A family $\mathcal{B} = \{x_n : n \in \mathbb{N}\}$ of vectors in a Hilbert space $\mathbb{H}$ is a *Schauder basis* for $\mathbb{H}$ if there exists a unique dual sequence $\{y_n : n \in \mathbb{N}\} \subset \mathbb{H}$ such that for every $x \in \mathbb{H}$,

$$
\lim_{N \to \infty} \sum_{n=1}^{N} \langle x, y_n \rangle x_n = x \quad \text{(norm convergence)}.
$$

**Ordering of the system**

The Schauder basis convergence may not be unconditional so the ordering of the system becomes important.
The Muckenhoupt $A_2$-class

A measurable, 1-periodic function $w : \mathbb{R} \to (0, \infty)$ is an $A_2(\mathbb{T})$-weight provided that

$$[w]_{A_2} := \sup_{I \in \mathcal{I}} \left( \frac{1}{|I|} \int_I w(\xi) \, d\xi \right) \left( \frac{1}{|I|} \int_I w(\xi)^{-1} \, d\xi \right) < \infty,$$

where $\mathcal{I}$ is the collection of intervals (arcs) on $\mathbb{T}$.

Proposition [Sikic and N., ACHA (2008)]

Let $\psi \in L_2(\mathbb{R}) \setminus \{0\}$. The system $B := \{\psi(\cdot - k) : k \in \mathbb{Z}\}$ forms a Schauder basis for $S(\psi)$, with $\mathbb{Z}$ ordered the natural way as $0, 1, -1, 2, -2, \ldots$, if and only if the periodization function $p_\psi$ satisfies the $A_2(\mathbb{T})$ condition.

Remark

- The result is based on the well-known Hunt-Muckenhoupt-Wheeden Theorem.
- Similar results for Gabor systems were obtained by Heil and Powell [J. Math. Phys. (2006)].
- The PSI result can be generalized to multivariate FSI spaces using a theory of product $A_2$-matrix weights [N., JFAA (2010)].
Conditional Schauder bases of integer translates

Examples

- Define \( \psi \in L_2(\mathbb{R}) \) by

\[
\hat{\psi}(\xi) = \sqrt{\ln \left( \ln(2 + |\xi|^{-1}) \right)} \cdot \chi_{[0,1)}(\xi).
\]

It follows that \( p_\psi(\xi) = \ln \left( \ln(2 + |\xi|^{-1}) \right), \xi \in [-1/2, 1/2] \). A direct calculation shows that \( p_\psi \in A_2(\mathbb{T}) \), so

\( B := \{ \psi(\cdot - k) : k \in \mathbb{Z} \} \) forms a Schauder basis for \( S(\psi) \).

However, \( p_\psi \) is not bounded and consequently \( B \) fails to be an unconditional Riesz basis for \( S(\psi) \).

- Another example is provided by \( \psi \in L_2(\mathbb{R}) \) defined by

\[
\hat{\psi}(\xi) = |\xi|^\alpha \cdot \chi_{[0,1)}(\xi),
\]

with \( \alpha \in (-1/2, 1/2) \).
The $A_2$ class is closely related to the functions of bounded mean oscillation.

**Definition**

Let $f \in L^1_{1,\text{loc}}(\mathbb{R})$ be 1-periodic, and let $\mathcal{I}$ be the collection of intervals (arcs) on $\mathbb{T}$. We say that $f \in BMO(\mathbb{T})$ provided that

$$
\|f\|_{BMO(\mathbb{T})} := \sup_{I \in \mathcal{I}} \frac{1}{|I|} \int_I |f(x) - f_I| \, dx < \infty,
$$

where $f_I := \frac{1}{|I|} \int_I f(x) \, dx$.

- One can verify that $\log(A_2(\mathbb{T})) \subset BMO(\mathbb{T})$.
- Conversely, for $f \in BMO(\mathbb{T})$ there is some $\alpha > 0$ such that $e^{\alpha f} \in A_2(\mathbb{T})$ [by the John-Nirenberg inequality].
- It is also easy to check that $L^\infty(\mathbb{T}) \hookrightarrow BMO(\mathbb{T})$. 
All of this is related to stability of integer translates by the fact that

\[ \{\psi(\cdot - k) : k \in \mathbb{Z}\} \text{ forms a Riesz basis } \iff \log(p_\psi) \in L_\infty \]

Question

Can we use the distance to \( L_\infty \) of \( \log(p_\psi) \in BMO(\mathbb{T}) \) to quantify the “quality” of a conditional Schauder basis?

Distance to \( L_\infty \)

For \( f \in BMO(\mathbb{T}) \) we let

\[ \text{dist}(f, L_\infty(\mathbb{T})) := \inf_{g \in L_\infty(\mathbb{T})} \| f - g \|_{BMO(\mathbb{T})}. \]
One additional observation

It is known that $L_\infty$ is not a closed subset of $BMO$. In fact,

$$\{ f \in BMO(\mathbb{T}) : \text{dist}(f, L_\infty) = 0 \} = \{ f \in BMO : e^{mf} \in A_2, m \in \mathbb{Z} \}.$$ 

This follows from the celebrated result by Garnett and Jones that asserts that $\text{dist}(f, L_\infty)$ and

$$\varepsilon(f) := \inf \{ \lambda > 0 : [e^{f/\lambda}]_{A_2(\mathbb{T})} < \infty \}$$

are in fact equivalent independent of $f \in BMO(\mathbb{T})$.

**Theorem (Garnett and Jones)**

There exist positive constants $C_1$ and $C_2$ such that for $f \in BMO(\mathbb{T})$,

$$C_1 \varepsilon(f) \leq \text{dist}(f, L_\infty(\mathbb{T})) \leq C_2 \varepsilon(f).$$
An example of an unbounded BMO function in \( \{ f : \text{dist}(f, L_{\infty}) = 0 \} \) is given by

\[
f(x) = \ln \left( \ln(2 + |x|^{-1}) \right), \quad x \in \mathbb{T}.
\]

This is a consequence of the fact that \( \ln^N(2 + |x|^{-1}) \in A_2(\mathbb{T}) \) for any \( N \in \mathbb{N} \), which follows by direct calculation.
Improved stability: The coefficient space

Let $B = \{x_n\}_{n \in \mathbb{N}}$ be a Schauder basis for $\mathbb{H}$ with dual system $\{y_n\}_{n \in \mathbb{N}}$. The coefficient space associated with $B$ is the sequence space given by

$$C(B) := \{\{\langle x, y_n \rangle\}_{n \in \mathbb{N}} : x \in \mathbb{H}\}.$$ 

**Controlling $C(B)$**

For a Riesz basis $B$, we have

$$C(B) = \ell_2.$$ 

For a normalized conditional Schauder basis $B$ in $\mathbb{H}$ one can find $2 \leq p < \infty$ (possibly very large) such that

$$C(B) \hookrightarrow \ell_p.$$ 

[Gurari˘ı and Gurari˘ı, 1971]
Theorem [Šikić and N. JFA (2014)]

Let $\psi \in L_2(\mathbb{R})$ and suppose that $p_\psi \in A_2(\mathbb{T})$. We let $C(E)$ denote the coefficient space for the Schauder basis $E = \{\psi(\cdot - k)\}_k$ for $S(\psi)$. Define $\varepsilon = \varepsilon(\ln p_\psi) := \inf\{\lambda > 0 : [p_\psi^{1/\lambda}]_{A_2} < \infty\}$. Then the following inclusion holds

$$
C(E) \subset \bigcap_{p_0 < p < \infty} \ell_p(\mathbb{Z}), \quad p_0 := \frac{2}{1 - \varepsilon}.
$$

In particular, if $\text{dist}(\ln(p_\psi), L_\infty(\mathbb{T})) = 0$ then

$$
C(E) \subset \bigcap_{2 < p < \infty} \ell_p(\mathbb{Z}).
$$
i. The $A_2$ condition implies that $L^2(\mathbb{T}, p_\psi) \hookrightarrow L^1(\mathbb{T})$

ii. Take $f = \lim_{N \to \infty} \sum_{|k| \leq N} \langle f, \tilde{\psi}(\cdot - k) \rangle \psi(\cdot - k) \in S(\psi)$ and let $m_f = J_{\psi}^{-1}(f) \in L^2(\mathbb{T}, p_\psi)$.

iii. Using i., verify that $m_f = \sum_{k \in \mathbb{Z}} \langle m_f, e_k \rangle_{L^2(\mathbb{T})} e^{2\pi ikx}$.

iv. Now use the Reverse Hölder Inequality for $p_\psi$ and the Hölder inequality to estimate

$$\|m_f\|_{L^r}$$

for $r \approx 2$.

v. Conclude using the Hausdorff-Young inequality.
Recall the previous example with $\psi \in L_2(\mathbb{R})$ defined by

$$\hat{\psi}(\xi) = \sqrt{\ln \left( \ln(2 + |\xi|^{-1}) \right)} \cdot \chi_{[0,1)}(\xi),$$

and $p_\psi(\xi) = \ln \left( \ln(2 + |\xi|^{-1}) \right)$, $\xi \in [-1/2, 1/2)$.

A direct calculation shows that $p_\psi^N \in A_2(\mathbb{T})$ for any $N \in \mathbb{N}$, so $\mathcal{E} = \{\psi(\cdot - k)\}_k$ forms a conditional Schauder basis for $S(\psi)$ with coefficient space for $\mathcal{E}$ controlled by

$$C(\mathcal{E}) \subset \bigcap_{2 < p < \infty} \ell_p(\mathbb{Z}).$$
Another point of view: Improved conditioning of Schauder bases

For a Schauder basis \( \mathcal{B} = \{ x_n : n \in \mathbb{N} \} \) in \( H \) with dual sequence \( \{ y_n : n \in \mathbb{N} \} \subset H \), we consider the partial sum operators
\[
S_N(x) = \sum_{n=1}^{N} \langle x, y_n \rangle x_n.
\]
The basis constant for \( \mathcal{B} \) is given by
\[
\kappa(\mathcal{B}) := \sup_{N \in \mathbb{N}} \| S_N \|.
\]

Theorem [Šikić and N. JFA (2014)]

Let \( \psi \in L_2(\mathbb{R}) \) with periodization function \( p_\psi \in A_2(\mathbb{T}) \). Suppose \( p_\psi \) satisfies \( \text{dist}(\ln p_\psi, L_\infty) = 0 \). Let \( \mathcal{E} = \{ \psi(\cdot - k) \}_k \). Then

i. If \( \ln p_\psi \in L_\infty(\mathbb{T}) \) then \( \mathcal{E} \) forms a Riesz basis for \( S(\psi) \).

ii. If \( \ln p_\psi \notin L_\infty(\mathbb{T}) \) then for every \( \eta > 0 \) there exists \( b \in L_\infty(\mathbb{T}) \) such that \( \tilde{\mathcal{E}} = \{ \varphi(\cdot - k) \}_k \), with \( \hat{\varphi} := \hat{\psi}_{eb} \), forms a Schauder basis for \( S(\psi) \) with Schauder basis constant at most \( 3 + O(\eta) \). The Schauder bases \( \mathcal{E} \) and \( \tilde{\mathcal{E}} \) are equivalent.
R. Hunt, B. Muckenhoupt, and R. Wheeden. 
Weighted norm inequalities for the conjugate function and Hilbert transform. 

The distance in BMO to $L^\infty$. 

M. Nielsen and H. Šikić. 
Schauder bases of integer translates. 

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On stability of finitely generated shift-invariant systems. 

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On stability of Schauder bases of integer translates. 