Multi-D wavelet construction using Quillen-Suslin theorem for Laurent polynomials

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Fourier Talks
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Outline

1. Review on Quillen-Suslin theorem and wavelet construction
2. Our new approaches for non-redundant wavelet construction
Outline

1. Review on Quillen-Suslin theorem and wavelet construction

2. Our new approaches for non-redundant wavelet construction
Quillen-Suslin theorem for Laurent polynomials

A column \(q\)-vector \(D(z)\) with Laurent polynomial entries is **unimodular** if it has a left inverse, i.e. if there exists a row \(q\)-vector \(F(z)\) s.t. \(F(z)D(z) = 1\).

We assume \(z \in \mathbb{C}^n, |z| = 1\).

Example:

\[
D(z) = \begin{bmatrix}
\frac{1}{2}, & \frac{1}{4}z_1^{-1} + \frac{1}{4}, & \frac{1}{4}z_2^{-1} + \frac{1}{4}, & \frac{1}{4}z_1^{-1}z_2^{-1} + \frac{1}{4}
\end{bmatrix}^T
\]

is unimodular since \([2, 0, 0, 0]\) is a left inverse of \(D(z)\).

Another left inverse of \(D(z)\) is

\[
\begin{bmatrix}
-\frac{1}{8}z_1^{-1} - \frac{1}{8}z_2^{-1} - \frac{1}{8}z_1^{-1}z_2^{-1} + \frac{5}{4} - \frac{1}{8}z_1 - \frac{1}{8}z_2 - \frac{1}{8}z_1z_2, & \frac{1}{4} + \frac{1}{4}z_1, & \frac{1}{4} + \frac{1}{4}z_2, & \frac{1}{4} + \frac{1}{4}z_1z_2
\end{bmatrix}
\]

The first one is simpler but the second one has better accuracy.

Theorem (Quillen-Suslin Thm for Laurent poly by Swan, 1978)

*Let \(D(z)\) be a unimodular column \(q\)-vector. Then there exists an invertible \(q \times q\) matrix \(T(z)\) s.t. \(T(z)D(z) = [1, 0, \ldots, 0]^T\).*
A column \( q \)-vector \( D(z) \) with Laurent polynomial entries is \textbf{unimodular} if it has a left inverse, i.e. if there exists a row \( q \)-vector \( F(z) \) s.t. \( F(z)D(z) = 1 \).

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**Theorem (Quillen-Suslin Thm for Laurent poly by Swan, 1978)**

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Review on Quillen-Suslin theorem and wavelet construction
Our new approaches for non-redundant wavelet construction

Quillen-Suslin theorem for Laurent polynomials

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We assume $z \in \mathbb{C}^n$, $|z| = 1$.

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\end{bmatrix}^T$

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Theorem (Quillen-Suslin Thm for Laurent poly by Swan, 1978)

Let $D(z)$ be a unimodular column $q$-vector. Then there exists an invertible $q \times q$ matrix $T(z)$ s.t. $T(z)D(z) = [1, 0, \ldots, 0]^T$. 

Multi-D wavelet construction using Quillen-Suslin for Laurent poly
**FB design problem:** Find

- $H(z), J_i(z), i = 1, \ldots, p - 1$: row $q$-vectors
- $D(z), K_i(z), i = 1, \ldots, p - 1$: column $q$-vectors s.t.

$$S(z)A(z) := \begin{bmatrix} D(z) & K_1(z) & \cdots & K_{p-1}(z) \end{bmatrix} \begin{bmatrix} H(z) \\ J_1(z) \\ \vdots \\ J_{p-1}(z) \end{bmatrix} = I_q$$

$A(z)$: analysis bank; $S(z)$: synthesis bank. The above identity is called the perfect reconstruction property. For the perfect reconstruction property to hold, we need $p \geq q$. 

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$$
\mathbf{S}(z) \mathbf{A}(z) := \begin{bmatrix}
\mathbf{D}(z) & \mathbf{K}_1(z) & \cdots & \mathbf{K}_{p-1}(z)
\end{bmatrix}
= \begin{bmatrix}
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\vdots \\
\mathbf{J}_{p-1}(z)
\end{bmatrix}
= \mathbf{I}_q
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The FB is called

- a non-redundant (or biorthogonal) FB if $p = q$. In this case, $A(z)$ and $S(z)$ are square matrices.
- a wavelet FB if
  - $H(z)$: lowpass row $q$-vector
  - $J_i(z), i = 1, \ldots, p - 1$: highpass row $q$-vectors
  - $D(z)$: lowpass column $q$-vector
  - $K_i(z), i = 1, \ldots, p - 1$: highpass column $q$-vectors

$\Rightarrow$ wavelet FB design: a key step in wavelet construction

- a wavelet FB with $m$ vanishing moments (VM) (for $m \geq 1$) if
  - $H(z)$: lowpass row $q$-vector
  - $J_i(z), i = 1, \ldots, p - 1$: row $q$-vectors with $m$ VM
  - $D(z)$: lowpass column $q$-vector
  - $K_i(z), i = 1, \ldots, p - 1$: column $q$-vectors with $m$ VM

$\Rightarrow$ leads to wavelets with $m$ VM (high performance)
Very brief introduction to wavelets

- Wavelets are a collection of functions obtained by scaling and translating a fixed set of functions (mother wavelets).
- Wavelet is a subfield of Harmonic analysis and highly interdisciplinary. Wavelets are used in many applications (e.g. image/signal processing, compressive sensing).

Examples (1-D): Haar (1909), VM=1; Daubechies (1987), VM=2

Constructing multi-D wavelets is challenging and important.
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Review on Quillen-Suslin theorem and wavelet construction
Our new approaches for non-redundant wavelet construction

Current approach
for designing non-redundant wavelet FBs

Theorem (by Chen-Han-Riemenschneider, 2000)

Suppose \( H(z), D(z) \) are lowpass vectors and

\[
\begin{bmatrix}
D(z) & K_1(z) & \cdots & K_{q-1}(z)
\end{bmatrix}
\begin{bmatrix}
H(z) \\
J_1(z) \\
\vdots \\
J_{q-1}(z)
\end{bmatrix} = I_q
\]

Then the following are equivalent.

1. \( H(z), D(z) \) have \( m \) accuracy (AC, or approximation order).
2. \( J_i(z), K_i(z), i = 1, \ldots, q - 1 \), have \( m \) VM.

Corollary (obtained by C-H-R and Q-S for Laurent polynomials)

Let \( H(z), D(z) \) be lowpass vectors with \( m \) AC and \( H(z)D(z) = 1 \).
Then there exist \( J_i(z), K_i(z), i = 1, \ldots, q - 1 \), with \( m \) VM such that the perfect reconstruction property holds.
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Corollary (obtained by C-H-R and Q-S for Laurent polynomials)

Let $H(z)$, $D(z)$ be lowpass vectors with $m$ AC and $H(z)D(z) = 1$. Then there exist $J_i(z)$, $K_i(z)$, $i = 1, \ldots, q - 1$, with $m$ VM such that the perfect reconstruction property holds.
Vanishing moments (VM) and accuracy (AC)

Assume the dilation is dyadic, and \( q = 2^n \).
Then, \( \Gamma := \{0, 1\}^n =: \{\nu_0 = 0, \nu_1, \ldots, \nu_{q-1}\} \) can be chosen.
Notice that \( \mathbb{Z}^n = \bigcup_{\nu \in \Gamma} (2\mathbb{Z}^n + \nu) \).

**Definition (for dyadic dilation case)**

For \( H(z) = [H_0(z), H_1(z), \ldots, H_{q-1}(z)] \), let \( H(z) = \sum_{j=0}^{q-1} z^{\nu_j} H_j(z^2) \).
Let \( m \) be a nonnegative integer. Then \( H(z) \) has

- **\( m \) VM** if \( \frac{\partial^k}{\partial \omega^k} H(e^{i\omega})|_{\omega=0} = 0, \forall |k| \leq m - 1 \), and
- **\( m \) AC** if \( \frac{\partial^k}{\partial \omega^k} H(e^{i\omega})|_{\omega=\gamma} = 0, \forall |k| \leq m - 1, \forall \gamma \in \{0, \pi\}^n \setminus 0 \).

VM and AC for column vector \( D(z) = [D_0(z), D_1(z), \ldots, D_{q-1}(z)]^T \)
is defined similarly by forming \( D(z) = \sum_{j=0}^{q-1} z^{-\nu_j} D_j(z^2) \).
\( H(z) \) or \( D(z) \) is the highpass vector iff it has \( \text{VM} \geq 1 \).
Current approach is not satisfactory

Finding $H(z)$, $D(z)$ satisfying assumptions of Corollary is not easy, especially if the AC or the spatial dimension $n$ is large.

Example: Let $n = 2$, and let

$$H(z) = \left[ \frac{1}{2}, \frac{1}{4}z_1^{-1} + \frac{1}{4}, \frac{1}{4}z_2^{-1} + \frac{1}{4}, \frac{1}{4}z_1^{-1}z_2^{-1} + \frac{1}{4} \right] : \text{lowpass with 2 AC.}$$

Then $[2, 0, 0, 0]^T: \text{lowpass, a right inverse of } H(z), \text{but with 0 AC.}$

Using Maple implementation of Algebraic Geometry theory (Cox-Little-O’Shea, 2006), we see any right inverse of $H(z)$ is

$$\begin{bmatrix}
2 \\
0 \\
0 \\
0 \\
\end{bmatrix} - \frac{1}{2}u_1(z) \begin{bmatrix}
z_1^{-1} + 1 \\
-2 \\
0 \\
0 \\
\end{bmatrix} - \frac{1}{2}u_2(z) \begin{bmatrix}
z_2^{-1} + 1 \\
0 \\
-2 \\
0 \\
\end{bmatrix} - \frac{1}{2}u_3(z) \begin{bmatrix}
z_1^{-1}z_2^{-1} + 1 \\
0 \\
0 \\
-2 \\
\end{bmatrix}$$

for some Laurent polynomials $u_1(z), u_2(z), u_3(z)$. To find a right inverse of $H(z)$ with 2 AC, one can use this parameterization. Usually done by fixing the total degree of Laurent poly $u_1, u_2, u_3$, and then increasing the total degree if needed (Riemenschneider-Shen, 1997; Han-Jia, 1999; Park, 2002).
Finding $H(z), D(z)$ satisfying assumptions of Corollary is not easy, especially if the AC or the spatial dimension $n$ is large.

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Then $[2, 0, 0, 0]^T$: lowpass, a right inverse of $H(z)$, but with 0 AC. Using Maple implementation of Algebraic Geometry theory (Cox-Little-O’Shea, 2006), we see any right inverse of $H(z)$ is

$$\begin{bmatrix}
2 \\
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\end{bmatrix} - \frac{1}{2}u_1(z) \begin{bmatrix}
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for some Laurent polynomials $u_1(z), u_2(z), u_3(z)$. To find a right inverse of $H(z)$ with 2 AC, one can use this parameterization. Usually done by fixing the total degree of Laurent poly $u_1, u_2, u_3$, and then increasing the total degree if needed (Riemenschneider-Shen, 1997; Han-Jia, 1999; Park, 2002).
Outline

1. Review on Quillen-Suslin theorem and wavelet construction

2. Our new approaches for non-redundant wavelet construction
Our approach (theory & algorithm) for designing non-redundant wavelet FBs

Inputs:
- $H(z)$: row, lowpass, $q$-vector with unimodularity, positive AC.
- $G(z)$: column, lowpass, $q$-vector with positive AC.
- $F(z)$: column, lowpass, $q$-vector, a right inverse of $H(z)$.

Algorithm: Set
- $D(z) := G(z) + F(z)(1 - H(z)G(z))$: column, $q$-vector
- $T(z)$: $q \times q$ invertible matrix s.t. $T(z)H(z)^T = [1, 0, \ldots, 0]^T$.
- $K_1(z), \ldots, K_{q-1}(z)$: 2nd to last columns of $T(z)^T$.
- $J_1(z), \ldots, J_{q-1}(z)$: 2nd to last rows of $T(z)^{-T}[I_q - F(z)H(z)][I_q - G(z)H(z)]$.

Output:
Wavelet FB: $(D(z), K_1(z), \ldots, K_{q-1}(z)), (H(z), J_1(z), \ldots, J_{q-1}(z))$
where $D(z)$ is a right inverse of $H(z)$ with positive AC.
Let $n = 2$.

Let $H(z) = G(z)^* = \left[ \frac{1}{2}, \frac{1}{4} z_1^{-1}, \frac{1}{4} z_2^{-1}, \frac{1}{4} z_1 z_2^{-1}, \frac{1}{4} \right]$; lowpass with 2 AC.

Let $F(z) = [2, 0, 0, 0]^T$; lowpass with $H(z)F(z) = 1$, but with 0 AC.

Set $D(z) = G(z) + F(z)(1 - H(z)G(z)) = H(z)^* + (1 - H(z)H(z)^*)F(z)$

$$D(z) = \left[ \begin{array}{c}
\frac{1}{2} \\
\frac{1}{4} + \frac{1}{4} z_1 \\
\frac{1}{4} + \frac{1}{4} z_2 \\
\frac{1}{4} + \frac{1}{4} z_1 z_2
\end{array} \right] + \left( -\frac{1}{16} z_1^{-1} - \frac{1}{16} z_2^{-1} - \frac{1}{16} z_1^{-1} z_2^{-1} + \frac{3}{8} - \frac{1}{16} z_1 - \frac{1}{16} z_2 - \frac{1}{16} z_1 z_2 \right) \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix}
-\frac{1}{8} z_1^{-1} & -\frac{1}{8} z_2^{-1} & -\frac{1}{8} z_1^{-1} z_2^{-1} & \frac{5}{4} - \frac{1}{8} z_1 - \frac{1}{8} z_2 - \frac{1}{8} z_1 z_2, \\
\frac{1}{4} + \frac{1}{4} z_1, \\
\frac{1}{4} + \frac{1}{4} z_2, \\
\frac{1}{4} + \frac{1}{4} z_1 z_2
\end{bmatrix}^T$$

We see that $D(z)$ has 2 AC.

From the implementation of Quillen-Suslin Theorem by Maple, we see that

$$T(z) := \begin{bmatrix}
2 & 0 & 0 & 0 \\
-\frac{1}{2} z_1^{-1} - \frac{1}{2} & 1 & 0 & 0 \\
-\frac{1}{2} z_2^{-1} - \frac{1}{2} & 0 & 1 & 0 \\
-\frac{1}{2} z_1^{-1} z_2^{-1} - \frac{1}{2} & 0 & 0 & 1
\end{bmatrix}$$

satisfies $T(z)H(z)^T = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Hence, we can get $K_1(z), K_2(z), K_3(z)$ from 2nd to 4th columns of $T(z)^T$ and $J_1(z), J_2(z), J_3(z)$ from 2nd to 4th rows of $T(z)^{-T} \left[ I_4 - F(z)H(z) \right] \left[ I_4 - G(z)H(z) \right]$. 

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Multi-D wavelet construction using Quillen-Suslin for Laurent poly
Example: Designing a 2-D wavelet FB

More precisely, the analysis bank $A(z)$ and the synthesis bank $S(z)$ are given as

$$A(z) = \begin{bmatrix} H(z) \\ J_1(z) \\ J_2(z) \\ J_3(z) \end{bmatrix}, \quad S(z) = \begin{bmatrix} D(z) & K_1(z) & K_2(z) & K_3(z) \end{bmatrix}$$

where

$$J_1(z) = \begin{bmatrix} -\frac{1}{8} & -\frac{1}{8} z_1 & -\frac{1}{16} z_1^{-1} & -\frac{7}{8} & -\frac{1}{16} z_2 & -\frac{1}{16} z_2^{-1} z_1 & -\frac{1}{16} z_1^{-1} & -\frac{1}{16} z_2^{-1} z_1 - \frac{1}{16} z_1^{-1} z_2 - \frac{1}{16} z_2^{-1} - \frac{1}{16} z_1 \end{bmatrix}$$

$$J_2(z) = \begin{bmatrix} -\frac{1}{8} & -\frac{1}{8} z_2 & -\frac{1}{16} z_1^{-1} z_2 & -\frac{1}{16} z_2 & -\frac{1}{16} z_2^{-1} z_1 & \frac{7}{8} - \frac{1}{16} z_2 & -\frac{1}{16} z_1^{-1} z_2 - \frac{1}{16} z_1^{-1} - \frac{1}{16} z_2 \\
-\frac{1}{8} & -\frac{1}{8} z_1 z_2 & -\frac{1}{16} z_1^{-1} z_2 - \frac{1}{16} z_2 - \frac{1}{16} z_1 z_2 & -\frac{1}{16} z_2^{-1} z_1 - \frac{1}{16} z_1 z_2 & -\frac{1}{16} z_1^{-1} z_2^{-1} + \frac{7}{8} - \frac{1}{16} z_1 z_2 \end{bmatrix}$$

and

$$K_1(z) = \begin{bmatrix} -\frac{1}{2} z_1^{-1} & -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad K_2(z) = \begin{bmatrix} -\frac{1}{2} z_2^{-1} & -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad K_3(z) = \begin{bmatrix} -\frac{1}{2} z_1^{-1} z_2^{-1} & -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$
Given a positive integer \( m \), let
- \( H(z) \): row, lowpass, \( q \)-vector w/ unimodularity, \( m \) AC.
- \( G(z) \): column, lowpass, \( q \)-vector w/ \( m \) AC.
- \( F(z) \): column, lowpass, \( q \)-vector, a right inverse of \( H(z) \).

Then one has an algorithm to construct a non-redundant wavelet FB with at least \( m \) VM so that

\[
\begin{bmatrix}
D(z) & * & \cdots & *
\end{bmatrix}
\begin{bmatrix}
H(z) \\
* \\
\vdots \\
* 
\end{bmatrix} = \mathbb{I}_q,
\]

where \( D(z) \), still determined by \( H(z) \), \( G(z) \) and \( F(z) \), is a right inverse of \( H(z) \) with at least \( m \) AC.
Summary

**Summary of the talk**

- Our method can be used to design non-redundant wavelet FBs for any dimension (it is especially useful for multi-D).
- It provides an algorithm for constructing a wavelet FB with \( m \) VM, starting from a unimodular lowpass vector with \( m \) AC.
- It does not require the initial unimodular lowpass vector to satisfy any additional assumption other than AC condition.

**Things that I did not cover in the talk**

- Our approaches work for any dilation.
- Connection of our method to Laplacian pyramid algorithms.
Remaining challenges

- Our approaches so far have been mostly algebraic, hence questions that are analytic in nature need to be answered separately.
- Currently we are using only the implementation of the big theorem (Quillen-Suslin Theorem for Laurent polynomials). We’ll try to fully exploit the powerfulness of the big theorem.
- Currently we are concerned with only the VM of wavelets. We’ll try to incorporate other properties such as symmetry, interpolatory property, and fast algorithms.
References

Thank you for your attention
Appendix

Outline

3  Appendix

Multi-D wavelet construction using Quillen-Suslin for Laurent poly
[z, z^2] is unimodular in Laurent polynomial ring since
\[ \left[ \frac{1}{2}z^{-1}, \frac{1}{2}z^{-2} \right]^T \] is a right inverse, but not in polynomial ring since there are no polynomials \( f(z), g(z) \) s.t. \( f(z)z + g(z)z^2 = 1 \).
Filter bank (FB)

$h, d, j_i, k_i : \mathbb{Z}^n \rightarrow \mathbb{R}, i = 1, \ldots, p - 1$, are filters (w/ finite supports) 
Downsampling & upsampling, with $n \times n$ sampling matrix $\Lambda$  
(w/ integers and all eigenvalues have magnitude larger than 1):

\[
y_{\downarrow}(k) = y(\Lambda k), \quad k \in \mathbb{Z}^n.
\]
\[
y_{\uparrow}(k) = \begin{cases} y(\Lambda^{-1} k), & k \in \Lambda \mathbb{Z}^n, \\ 0, & \text{otherwise}. \end{cases}
\]

i.e. for $n = 1, \Lambda = 2$, $y = (\ldots, y(-1), y(0), y(1), \ldots)$, we have
\[
y_{\downarrow} = (\ldots, y(-2), y(0), y(2), \ldots)
\]
\[
y_{\uparrow} = (\ldots, y(-1), 0, y(0), 0, y(1), \ldots).
\]

Filter bank (FB) problem is to find \(\{h, j_1, \ldots, j_{p-1}\}, \{d, k_1, \ldots, k_{p-1}\}\) 
s.t. \(d \ast ((h \ast x)_{\downarrow})_{\uparrow} + \sum_{i=1}^{p-1} k_i \ast ((j_i \ast x)_{\downarrow})_{\uparrow} = x\), for any finitely supported signal $x$. 

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Multi-D wavelet construction using Quillen-Suslin for Laurent poly
Laurent polynomial lowpass/highpass vectors

\( \Lambda: n \times n \) sampling matrix

\( q = | \det \Lambda | \)

\( \Gamma: \) a set of representatives of distinct cosets of \( \mathbb{Z}^n / \Lambda \mathbb{Z}^n \) with 0.

Then \( \Gamma =: \{ \nu_0 = 0, \nu_1, .., \nu_{q-1} \}, \mathbb{Z}^n = \bigcup_{\nu \in \Gamma} (\Lambda \mathbb{Z}^n + \nu) \).

**Definition**

Let \( H(z) = [H_0(z), H_1(z), \ldots, H_{q-1}(z)] \) be a row \( q \)-vector.

- \( H(z) \) is the (polyphase) **lowpass** vector if \( H(e^{i\omega})|_{\omega=0} = \sqrt{q} \)
- \( H(z) \) is the (polyphase) **highpass** vector if \( H(e^{i\omega})|_{\omega=0} = 0 \)

where

\[
H(z) = \sum_{j=0}^{q-1} z^{\nu_j} H_j(z^\Lambda)
\]

The type of column \( q \)-vector \( D(z) = [D_0(z), D_1(z), \ldots, D_{q-1}(z)]^T \) is defined similarly by forming \( D(z) = \sum_{j=0}^{q-1} z^{-\nu_j} D_j(z^\Lambda) \).
Our theory: full version

Theorem (by Hur-Park-Zheng)

Let

- $\alpha_H, \beta_H, \alpha_G, \beta_G > 0$ and $\alpha_F \geq 0$: integers.
- $H(z)$: unimodular lowpass row $q$-vector w/ $\alpha_H$ AC, $\beta_H$ FL.
- $G(z)$: lowpass column $q$-vector w/ $\alpha_G$ AC and $\beta_G$ FL.
- $F(z)$: lowpass column $q$-vector w/ $\alpha_F$ AC, $H(z)F(z) = 1$.

Then one can construct a non-redundant wavelet $FB$ so that

$$\begin{bmatrix} H(z) \\ * \\ \vdots \\ * \end{bmatrix} \ast \cdots \ast \begin{bmatrix} D(z) \\ * \\ \vdots \\ * \end{bmatrix} = I_q, \quad D(z) := G(z) + F(z)(1 - H(z)G(z))$$

with lowpass $D(z)$ w/ at least $\min\{\alpha_G, \alpha_F + \beta_G, \alpha_F + \beta_H\} > 0$ AC.

FL: flatness
Our algorithm to get a wavelet FB: full version

Inputs:
- $\alpha_H, \beta_H, \alpha_G, \beta_G > 0$ and $\alpha_F \geq 0$: integers.
- $H(z)$: unimodular lowpass row $q$-vector w/ $\alpha_H$ AC, $\beta_H$ FL.
- $G(z)$: lowpass column $q$-vector w/ $\alpha_G$ AC and $\beta_G$ FL.
- $F(z)$: lowpass column $q$-vector w/ $\alpha_F$ AC, $H(z)F(z) = 1$.

Output:
- wavelet FB whose lowpass row vector is $H(z)$.

Procedure:

Step 1 Set $D(z) := G(z) + F(z)(1 - H(z)G(z))$.
Step 2 Find invertible $K(z)$ s.t. $K(z)H(z)^T = [1, 0, .., 0]^T$.
Step 3 Let $K_1(z), .., K_{q-1}(z)$ be the 2nd to last columns of $K(z)^T$.
Step 4 Let $J_1(z), .., J_{q-1}(z)$ be the 2nd to last rows of
$K(z)^{-T}[I_q - F(z)H(z)][I_q - G(z)H(z)]$.
Our algorithm to get a wavelet FB w/ at least $\alpha_H$ VM: full version

**Inputs:**
- $\alpha_H, \beta_H > 0$ and $\alpha_F \geq 0$: integers.
- $H(z)$: unimodular lowpass row $q$-vector w/ $\alpha_H$ AC, $\beta_H$ FL.
- $F(z)$: lowpass column $q$-vector w/ $\alpha_F$ AC, $H(z)F(z) = 1$.

**Output:**
- wavelet FB w/ lowpass row vector $H(z)$ and at least $\alpha_H$ VM.

**Procedure:**

**Step 1** Initialize $Iter := 1$ and $D(z) := H(z)^* + F(z)(1 - H(z)H(z)^*)$

**Step 2** While $(\alpha_F + (Iter)\beta_H < \alpha_H)$
- $Iter := Iter + 1$; $D(z) := H(z)^* + D(z)(1 - H(z)H(z)^*)$

**Step 3** Find invertible $K(z)$ s.t. $K(z)H(z)^T = [1, 0, .., 0]^T$.

**Step 4** Define $K_1(z), .., K_{q-1}(z)$ and $J_1(z), .., J_{q-1}(z)$ as previous.
Vanishing moments (VM), flatness (FL), accuracy (AC)

**Definition**

For \( H(z) = [H_0(z), H_1(z), \ldots, H_{q-1}(z)] \), let \( H(z) = \sum_{j=0}^{q-1} z^j H_j(z^\Lambda) \).

Let \( m \) be a nonnegative integer.

- \( H(z) \) has \( m \) **VM** if \( \frac{d^k}{d\omega^k} H(e^{i\omega})|_{\omega=0} = 0, \forall |k| \leq m - 1 \)
- \( H(z) \) has \( m \) **FL** if \( \frac{d^k}{d\omega^k} (\sqrt{q} - H(e^{i\omega}))|_{\omega=0} = 0, \forall |k| \leq m - 1 \)
- \( H(z) \) has \( m \) **AC** if \( \frac{d^k}{d\omega^k} H(e^{i\omega})|_{\omega=\gamma} = 0, \forall |k| \leq m - 1, \forall \gamma \in \Gamma^* \setminus \{0\} \)

\( \Gamma^* \): set of rep. of distinct cosets of \( 2\pi(((\Lambda^T)^{-1}\mathbb{Z}^n)/\mathbb{Z}^n) \) w/ 0.

VM, FL, AC for column vector \( D(z) = [D_0(z), D_1(z), \ldots, D_{q-1}(z)]^T \)
is defined similarly by forming \( D(z) = \sum_{j=0}^{q-1} z^{-j} D_j(z^\Lambda) \).

\( H(z) \) or \( D(z) \) is the highpass vector iff it has **VM** \( \geq 1 \)
\( H(z) \) or \( D(z) \) is the lowpass vector iff it has **FL** \( \geq 1 \)
Equivalent conditions for VM, FL, and AC

Theorem

For $\mathbb{H}(z) = [H_0(z), H_1(z), \ldots, H_{q-1}(z)]$, let $H(z) = \sum_{j=0}^{q-1} z^{\nu_j} H_j(z^\Lambda)$. Let $m$ be a nonnegative integer.

- $\mathbb{H}(z)$ has $m$ VM iff $H(e^{i\omega}) \approx O(|\omega|^m)$ (at $\omega = 0$)
- $\mathbb{H}(z)$ has $m$ FL iff $\sqrt{q} - H(e^{i\omega}) \approx O(|\omega|^m)$ (at $\omega = 0$)
- $\mathbb{H}(z)$ has $m$ AC iff $H(e^{i(\omega + \gamma)}) \approx O(|\omega|^m)$ (at $\omega = 0$), $\forall \gamma \in \Gamma^* \setminus 0$

$\Gamma^*$: set of rep. of distinct cosets of $2\pi((\Lambda^T)^{-1}\mathbb{Z}^n)/\mathbb{Z}^n)$ w/ 0.