ABSTRACT

Title of dissertation:	BEURLING WEIGHTED SPACES, PRODUCT-CONVOLUTION OPERATORS, AND THE TENSOR PRODUCT OF FRAMES
	Abdelkrim Bourouihiya, Doctor of Philosophy, 2006
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G. Gaudry solved the multiplier problem for Beurling algebras, i.e., he identified the space of all multipliers of a Beurling algebra with a weighted space of bounded measures. In the first part of this thesis, we solve multiplier problems for some Beurling weighted spaces. We identify the space of all multipliers of some Beurling weighted spaces with the dual of spaces of Figà-Talamanca type.

A paper by R.C Busby and H.A.Smith gives necessary and sufficient conditions for the compactness of product-convolution operators. In the second part of this thesis, we present some applications of the result of R.C Busby and H.A.Smith; and we prove that the eigenfunctions of certain product-convolution operators can be obtained as solutions of some differential equations. Incidentally, we obtain classical special functions as eigenfunctions of these product-convolution operators.

In the third part of this thesis, we prove that the tensor product of two sequences is a frame (Riesz basis) if and only if each part of this tensor product is a frame (Riesz basis). We use this result to extend the Lyubarskii and Seip-Wallstén

theorem, characterizing Gabor frames generated by the Gaussian function, to higher dimensions.

BEURLING WEIGHTED SPACES, PRODUCT-CONVOLUTION OPERATORS, AND THE TENSOR PRODUCT OF FRAMES

by

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DEDICATION

To the memory of my mother.

To my wife Sabah.

To my daughter Nora.

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TABLE OF CONTENTS

Chapter 1

Introduction and outline of thesis

1.1 Structure of thesis

This thesis deals with three topics in abstract harmonic analysis: Multipliers of Beurling Weighted Spaces (Chapter 2); Product-Convolution Operators (Chapter 3); and the Tensor Product of Frames (Chapter 4). The topics are independent of each other.

In order to provide background and an outline in this chapter, I shall use some definitions which are not fully explained until Chapters 2-4. However, there is a list of notation with short explanations in section 1.3. Further, there are two appendices listing relevant results from harmonic analysis on locally compact abelian groups and operator theory.

Section 1.2 is an outline of the results in the thesis, and it also provides some perspective. In particular, I have listed each of my original contributions. These are Theorem 2.24, Theorem 2.36, Proposition 3.12, Proposition 3.13, Theorem 4.8, Theorem 4.12, Lemma 4.9, Lemma 4.23, Theorem 4.26, and Corollary 4.27.

1.2 Outline and perpective

Through out this chapter, G is a locally compact abelian group and dx is a Haar measure on G.

A *Beurling weight* on G is a measurable locally bounded function ω satisfying, for each $x, y \in G$, the following two properties: $\omega(x) \geq 1$ and $\omega(x + y) \leq \omega(x)\omega(y)$. The spaces $L^p_\omega(G) := L^p(G, \omega dx)$ are called *Beurling weighted spaces.* It can be shown that $L^1_\omega(G)$ is a commutative Banach algebra for the convolution between functions, the so called *Beurling Algebra.* Beurling weighted spaces and precisely Beurling algebras were introduced by A.Beurling ([4], 1938). These algebras give important examples of standard Banach algebras that have played a central role in building and understanding spectral synthesis theory ([1], [2], [4], [9], [15], [25], [26], and [30]). In the last few years Beurling weighted spaces have started to appear in the the theory of time-frequency analysis and applied mathematics. T.Strohmer has used them to model some problems in mobile communications ([11], [17], [33], and [34]).

Let E and F be two Banach spaces of measurable functions, and assume that E and F are stable by translations. A *multiplier* $E \to F$ is a bounded operator commuting with all *translations.* We denote by $M(E, F)$ the space of all multipliers $E \to F$. In Chapter 2, I study multiplier problems for Beurling weighted spaces. The first two sections, 2.1-2.2, contain the necessary material to present my results in sections 2.3-2.5.

In section 2.1, I collect some results about multipliers of $L^p(G)$. In section 2.2, I define Beurling weighted spaces and give some of their properties. I emphasize the fact that ωdx is a positive measure having several common properties with dx, the Haar measure on the group G. However, I note that a translation operator is an isometry on $L^p(G)$, while it is not in general an isometry on $L^p_\omega(G)$. This fact is closely related to multiplier problems for $L^p_\omega(G)$.

In section 2.3, I present a new proof of a known result, due to G.Gaudry [15], stating that $M(L^1_\omega(G))$ can be identified with the weighted space of bounded measures $M_\omega(G) := \{\mu : \mu \text{ is a bounded measure and } \|\mu\|_\omega := \int \omega|\mu| < \infty\}$.

In section 2.4, I prove the following new result.

Theorem 2.24. *Let $T : L^1_\omega(G) \to L^p_\omega(G)$ be a bounded linear transformation, where $p > 1$. Then*

(i) $T \in M(L^1_\omega(G), L^p_\omega(G))$ if and only if there exists a unique function $g \in L^p_\omega(G)$ such that $T = T_g : f \to g * f$, $f \in L^1_\omega(G)$.

(ii) There exists a constant $c \geq 1$ dependent only on the weight function ω, such that

$$\| T_g \|_{1,p,\omega} \leq \| g \|_{p,\omega} \leq c \| T_g \|_{1,p,\omega} .$$

(iii) $M(L^1_\omega(G), L^p_\omega(G))$ and $L^p_\omega(G)$ are topologically and algebraically identified by the mapping of part (i).

There is no known identification of $M(L^p_\omega(G))$. In section 2.5, I show that the space $M(L^p_\omega(G))$ can be embedded in the space $M(L^p_\omega(G), L^p(G))$, i.e., there is a continuous linear injection $M(L^p_\omega(G)) \to M(L^p_\omega(G), L^p(G))$. To obtain a characterization of $M(L^p_\omega(G), L^p(G))$, I define a new space of Figà-Talamanca type. Let p' be such that $\frac{1}{p} + \frac{1}{p'} = 1$. I define the *weighted Figà-Talamanca space* $A^p_\omega(G)$ as follows:

$$A^p_\omega(G) =: \{\sum_{i=1}^{\infty} f_i * g_i : f_i, g_i \in \mathcal{C}_c(G) \ \ and \ \ \sum_{i=1}^{\infty} \|f_i\|_{p,\omega} \|g_i\|_{p'} < \infty\},$$

endowed with the norm

$$|||f|||_\omega = \inf\{\sum_{i=1}^{\infty} \|f_i\|_{p,\omega} \|g_i\|_{p'} : f_i, g_i \in \mathcal{C}_c(G) \ \ and \ \ f = \sum_{i=1}^{\infty} f_i * g_i\},$$

where $\mathcal{C}_c(G)$ is the space of continuous compactly supported functions on G.

First I prove that $A^p_\omega(G)$ is a Banach space. Then I prove the following new result.

> **Theorem 2.36.** *Let $p > 1$. There exists an isometric linear isomorphism of $M(L^p_\omega(G), L^p(G))$ into $(A^p_\omega(G))^*$, the Banach space of continuous linear functionals on $A^p_\omega(G)$.*

In [18], A.T.Gürkanli and S.Öztop considered the space $M(L^p_\omega(G), L^p_{\omega^{1-p'}}(G))$, $1 < p \leq 2$, and identified it with the dual of the Banach space,

$$'A^p_\omega(G) := \{\sum_{i=1}^{\infty} f_i * g_i : f_i, g_i \in \mathcal{C}_c(G) \ \ and \ \ \sum_{i=1}^{\infty} \|f_i\|_{p,\omega} \|g_i\|_{p',\omega} < \infty\},$$

endowed with the norm

$$'|||f|||_\omega = \inf\{\sum_{i=1}^{\infty} \|f_i\|_{p,\omega} \|g_i\|_{p',\omega} : f_i, g_i \in \mathcal{C}_c(G) \ \ and \ \ f = \sum_{i=1}^{\infty} f_i * g_i\}.$$

To avoid confusion, a left prime is added to these newly defined spaces. My result improves the Gürkanli and Öztop result [18], since

$$M(L^p_\omega(G)) \hookrightarrow (A^p_\omega(G))^* \hookrightarrow \ ('A^p_\omega(G))^*.$$

I also note that the techniques used by Gürkanli and Öztop and my techniques are completely different.

Except for trivial cases, the operators $C_f : g \to f * g$ and $M_\varphi : g \to \varphi g$ are never compact on $L^2(\mathbb{R})$. However, the composition of these two operators is, in some cases, compact. A paper by R.C Busby and H.A.Smith [6] gives necessary and sufficient conditions on $\varphi \in L^\infty(G)$ for the compactness of the *product-convolution operator* $M_\varphi C_f$, where $f \in L^1(G)$. In Chapter 3, I study some properties of product-convolution operators. Appendix B contains the essentials of operator theory needed for Chapter 3.

In section 3.1, I prove that if φ belongs to the closure of $L^p(G) \cap L^\infty(G)$ in $L^\infty(G)$ and if $f \in L^1(G)$, then the product-convolution operator $M_\varphi C_f$ is compact. This can also be deduced from the R.C Busby and H.A.Smith results [6]. My proof is based on approximations of compact operators by Hilbert-Schmidt operators and a property of C*-algebras. The proof of R.C Busby and H.A.Smith is based on properties of mixed norm spaces. In section 3.2, I present some applications of the results of section 3.1.

In section 3.3, I prove the following new result.

Proposition 3.12. *Let $h \in L^1(\mathbb{R})$, and $\varphi, \psi \in L^\infty(\mathbb{R})$. Let $x \in L^2(\mathbb{R})$, and assume that the functions φ and $h * \psi x$ are twice differentiable on an open set $\Omega \subset \mathbb{R}$. Consider on $L^2(\mathbb{R})$ the operator $H = M_\varphi C_h M_\psi$. Then*

(i) If x is an eigenfunction of the operator $H = M_\varphi C_h M_\psi$, associated with a characteristic value λ, then x is, on Ω, a solution of the integro-

differential equation:

$$E_\lambda: \ \varphi^2 y'' - 2\varphi'\varphi y' + (2(\varphi')^2 - \varphi''\varphi)y = \lambda\varphi^3(h * \psi y)''.$$

(ii) If x is a solution of E_λ, then $\lambda H(x) - x$ is a solution, on Ω, of the differential equation

$$E'_\lambda: \ \varphi^2 y'' - 2\varphi'\varphi y' + (2(\varphi')^2 - \varphi''\varphi)y = 0.$$

This new result is interesting, since it gives the spectral decomposition of some compact operators, which are not necessarily Hilbert-Schmidt operators. I also prove the following new result, which is useful for examples:

Proposition 3.13. *Let $h = e^{-|t|}$ and let $\varphi, \psi \in L^\infty(\mathbb{R})$. Assume that φ is twice differentiable on some open set $\Omega \subset \mathbb{R}$, and ψ is continuous on Ω. Consider on $L^2(\mathbb{R})$ the operator $H = M_\varphi C_h M_\psi$. If x is an eigenfunction of the operator H, associated with a characteristic value λ, then*

(i) the function x is twice differentiable on the open set Ω, and

(ii) The function x is, on Ω, a solution of the differential equation

$$\varphi^2 x'' - 2\varphi'\varphi x' + (2(\varphi')^2 - \varphi''\varphi - \varphi^2 + 2\lambda\varphi^3\psi)x = 0.$$

I end this section by some relevant examples. Among the consequences of these examples, I obtain special functions as eigenfunctions of product-convolution operators.

It is known that the tensor product of two orthonormal bases is an orthonormal basis. In Chapter 4, I prove the following new result.

Theorem 4.26. *The sequence* $(f_i^k)_{i\in\mathcal{I}_k}$ *is a frame (Riesz basis) for a Hilbert space* $\mathcal{H}_k$, $k \in \{1,2\}$, *if and only if* $(f_i^1 \otimes f_j^2)_{(i,j)\in\mathcal{I}_1\times\mathcal{I}_2}$ *is a frame (Riesz basis) for* $\mathcal{H}_1 \otimes \mathcal{H}_2$.

This result improves a result by C.Heil, J.Ramanathan, and P.Topiwala [19]. They prove that the tensor product of a frame with itself is a frame.

Section 4.1 and appendix B contain the essentials of operator theory needed for Chapter 4. I denote by $\mathcal{L}(X)$ the space of all bounded operators on a Banach space X. It is known that if $F_k \in \mathcal{L}(\mathcal{H}_k)$, $k \in \{1,2\}$, then $F_1 \otimes F_2 \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ [14]. In section 4.2, I prove two new results. The first new result can be stated as follows:

Theorem 4.8. *For each* $k \in \{1,2\}$, *let* $(F_N^k)_{N>0}$ *be a bounded sequence in* $\mathcal{L}(\mathcal{H}_k)$. *If, for each* $k \in \{1,2\}$, *the sequence* $(F_N^k)_{N>0}$ *converges in the strong operator topology to* $F^k \in \mathcal{L}(\mathcal{H}_k)$, *then* $(F_N^1 \otimes F_N^2)_{N>0}$ *converges in the strong operator topology to* $F^1 \otimes F^2$.

The second new result can be stated as follows:

Theorem 4.12. *The operator* $F_1 \otimes F_2$ *is invertible in* $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ *if and only the operator* F_k *is invertible in* $\mathcal{L}(\mathcal{H}_k)$ *for each* $k \in \{1,2\}$.

To prove my second result, I use the following new lemma.

Lemma 4.9. *For each* $k \in \{1,2\}$ *let* F_k *be a nonzero bounded operator on* $\mathcal{H}_k$, f_k *be a unit vector such that* $F_k(f_k) \neq 0$, $U_k : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_k$ *and* $V_k : \mathcal{H}_k \to \mathcal{H}_1 \otimes \mathcal{H}_2$, *defined for each* $f \in \mathcal{H}_1$, $g \in \mathcal{H}_2$, *and* $H \in$

$\mathcal{H}_1 \otimes \mathcal{H}_2$, by $U_1(H) = H(F_2(f_2))$, $V_1(f) = E_{f,f_2}$, $U_2(H) = H^*(F_1(f_1))$, and $V_2(g) = E_{f_1,g}$. Then

(i) $\| U_1 \|_{O(\mathcal{H}_1\otimes\mathcal{H}_2,\mathcal{H}_1)} \leq \| F_2(f_2) \|_2$, $\| U_2 \|_{O(\mathcal{H}_1\otimes\mathcal{H}_2,\mathcal{H}_2)} \leq \| F_1(f_1) \|_1$, and the operators V_1 and V_1 are isometric;

(ii) $U_1[F_1 \otimes F_2]V_1 = \| F_2(f_2) \|_2^2 F_1$ and $U_2[F_1 \otimes F_2]V_2 = \| F_1(f_1) \|_1^2 F_2$; and

(iii) $U_kV_k =< F_k(f_k), f_k > I_{\mathcal{H}_k}$, for each $k \in \{1, 2\}$.

Theorem 4.8, Lemma 4.9, and Theorem 4.12 are new contributions to the theory of the tensor product. Further I use them to prove Theorem 4.26.

In section 4.3, I define frames and state some of their properties. In section 4.4, I prove Theorem 4.26, the main new result obtained in this chapter. For the proof I use, in addition to all results obtained in section 4.2, the following new lemma.

Lemma 4.23. *Let $(f_n)_{n>0}$ be a sequence and let $(e_n)_{n>0}$ be an orthornormal basis in a Hilbert space $\mathcal{H}$. We define, formally, the linear operator*

$$F(x) = \sum_{n>0} < x, f_n > e_n.$$

(i) The sequence $(f_n)_{n>0}$ is a frame with frame bounds A and B if and only if the operator F is bounded and, for each $x \in \mathcal{H}$, we have

$$A \| x \|^2 \leq \| F(x) \|^2 \leq B \| x \|^2 .$$

(ii) $(f_n)_{n>0}$ is a Riesz basis if and only if the operator F is bijective.

This lemma is an interesting connection between the theory of frames and the theory of operators.

I end this chapter by an application. Let $g \in L^2(\mathbb{R}^d) \setminus \{0\}$ and let $\alpha, \beta > 0$. The Gabor system generated by g, α, and β is

$$\mathcal{G}(g, \alpha, \beta) = \{\, T_{\alpha m} M_{\beta n} g : m, n \in \mathbb{Z}^d \},$$

where $T_{\alpha m} M_{\beta n} g(x) = e^{2\pi i \beta n.(x - \alpha m)} g(x - \alpha m)$. The following result was conjectured by I. Daubechies and A. Grossmann [7] and then was proved independently by Y. Lyubarski [24] and K. Seip and R. Wallstén [32].

Lyubarskii and Seip-Wallstén Theorem. *Let $\varphi(x) = 2^{\frac{1}{4}} e^{-\pi x^2}$ be the Gaussian function on $\mathbb{R}$.*

$$\mathcal{G}(\varphi, \alpha, \beta) \ \ \textit{is a frame for} \ \ L^2(\mathbb{R}) \ \ \textit{if and only if} \ \ \alpha\beta < 1.$$

Using Theorem 4.26 and the Lyubarskii and Seip-Wallstén Theorem, I prove the following new result.

Corollary 4.27. *Let $\varphi(x) = 2^{\frac{d}{4}} e^{-\pi |x|^2}$ be the Gaussian function on $\mathbb{R}^d$.*

$$\mathcal{G}(\varphi, \alpha, \beta) \ \ \textit{is a frame for} \ \ L^2(\mathbb{R}^d) \ \ \textit{if and only if} \ \ \alpha\beta < 1.$$

This result is important, since there is only a handful of functions $g \in L^2(\mathbb{R}^d)$ for which the precise range of α, β, such that $\mathcal{G}(g, \alpha, \beta)$ is a frame, is known [17].

1.3 List of Notation

G, a locally compact abelian group.

dx, a Haar measure on G, i.e., a regular positive measure invariant by translations, see p 86.

$\widehat{G}$, the dual group of the group G, see p 87.

$\mathcal{C}_0(G)$, the space of continuous functions f vanishing at infinity, i.e., for all $\varepsilon > 0$ there exists a compact set $K \subset G$ such that $\mid f(x) \mid < \varepsilon$ for almost all $x \in G \setminus K$.

$\mathcal{C}_c(G)$, the space of continuous compactly supported functions.

$L^p(G)$, the space of function f such that the function $\mid f \mid^p$ is integrable.

$L^\infty(G)$, the space of essentially bounded functions.

$M(G)$, the space of bounded measures, see p 88.

$\mid \mu \mid$, the total variation of the measure μ, see p 89.

$\iota_a f(x) = f(x-a)$, the translation by a of f.

$f * g$, the convolution of f and g, i.e., $f * g(x) = \int f(x-y)g(y)dy$.

$M(E,F)$, the space of all multipliers: $E \to F$, i.e., the space of all bounded operators commuting with all translations.

$M(L^p(G), L^q(G))$, the space of all multipliers: $L^p(G) \to L^q(G)$.

$A^p(G)$, a Figà-Talamanca space, see p 14.

ω, a Beurling weight, see p 15.

$L^p_\omega(G)$, a Beurling weighted space, i.e., the space of function f such that $\int \mid f \mid^p \omega dx$ is finite.

$\parallel f \parallel_{p,\omega}$, the norm of each $f \in L^p_\omega(G)$.

$M_\omega(G)$, the space of bounded measures μ such that $\int \omega d \mid \mu \mid$ is finite.

$\parallel \mu \parallel_{p,\omega}$, the norm of each $\mu \in M_\omega(G)$, i.e., $\parallel \mu \parallel_{p,\omega} = \int \omega d \mid \mu \mid$.

$M(L^p_\omega(G), L^q_\omega(G))$, the space of all multipliers on $L^p_\omega(G) \to L^q_\omega(G)$.

$\parallel T \parallel_{p,q,\omega}$, the norm of each $T \in M(L^p_\omega(G), L^q_\omega(G))$.

$A^p_\omega(G)$, a weighted space of Figà-Talamanca type, see p 36.

C_f, a convolution operator, i.e., $C_f : g \to f * g$.

M_φ, a multiplication operator, i.e., $M_\varphi : g \to \varphi g$.

$\mathcal{H}$, a Hilbert space.

$< f, g >$, the inner product of $f, g \in \mathcal{H}$.

$\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, the space of all bounded operators: $\mathcal{H}_1 \to \mathcal{H}_2$.

$\mathcal{LC}(\mathcal{H}_1, \mathcal{H}_2)$ the space of all compact operators: $\mathcal{H}_1 \to \mathcal{H}_2$, see p 90.

$\mathcal{L}_2(\mathcal{H}_1, \mathcal{H}_2)$, the space of all Hilbert-Schmidt operators: $\mathcal{H}_1 \to \mathcal{H}_2$, see p 66.

$E_{f,g}$, a rank one operator, i.e., $E_{f,g} : x \to < x, g > f$.

$\parallel T \parallel_{O(\mathcal{H}_1,\mathcal{H}_2)}$, the norm of each $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$.

$\parallel T \parallel_{H(\mathcal{H}_1,\mathcal{H}_2)}$, the norm of each $T \in \mathcal{L}_2(\mathcal{H}_1, \mathcal{H}_2)$.

$\mathcal{H}_1 \otimes \mathcal{H}_2$, the tensor product of two Hilbert spaces, see p 67.

$S \otimes T$, the tensor product of a bounded operator S on $\mathcal{H}_1$ and a bounded operator T on $\mathcal{H}_2$, see p 68.

$\mathcal{G}(g, \alpha, \beta)$, a Gabor frame generated by g, α, and β, see p 77.

Chapter 2

Multipliers of Beurling weighted spaces

In section 2.1, I list some results about *multipliers* of $L^p(G)$ spaces. In section 4.2, I define *Beurling weighted spaces* and give some of their properties. In section 4.3, I present a new proof of a known result, due to G.Gaudry [15], stating that $M(L^1_\omega(G))$, the space of all multipliers of $L^1_\omega(G)$, can be identified with the *weighted space of bounded measures* $M_\omega(G)$. In section 4.4, I prove a new result, viz., the identification of $M(L^1_\omega(G), L^p_\omega(G))$, the space of all multipliers $L^1_\omega(G) \to L^p_\omega(G)$, with the space $L^p_\omega(G)$. There is no known identification of $M(L^p_\omega(G))$, the space of all multipliers of $L^p_\omega(G)$. In section 2.5, I show that the space $M(L^p_\omega(G))$ can be embedded in $M(L^p_\omega(G), L^p(G))$, the space of all multipliers $L^p_\omega(G) \to L^p(G)$. To obtain a characterization of $M(L^p_\omega(G), L^p(G))$, I define $A^p_\omega(G)$, a new space of *Figà-Talamanca type*. Then I prove the isometric identification of the space $M(L^p_\omega(G), L^p(G))$ with the dual of $A^p_\omega(G)$. I end this section by showing how the main result of section 2.5 improves a result due to A. Gürkanli and S. Öztop [18]. Section 2.6, is a summary of all results of this chapter. The measure and operator theory results needed as background for this chapter can be found in appendices A and B.

Through out this chapter G is a *locally compact abelian group* and dx is a *Haar measure* on G. We denote by $\widehat{G}$ the *dual group* of G. The unit of G shall be denoted by e. The *translation* by $a \in G$ of a measurable function f is defined by the formula $\iota_a f(x) = f(x-a)$. We denote by $\mathcal{C}_0(G)$ the space of continuous functions vanishing at infinity and by $\mathcal{C}_c(G)$ the space of continuous compactly supported functions. If $1 \leq p < \infty$, $L^p(G)$ shall denote the space of functions f such that $\mid f \mid^p$ is integrable. We denote by $\widehat{f}$ the *Fourier transform* of an integrable function f on G, see appendix A for detailed definitions.

2.1 Multipliers of $L^p(G)$

All results presented in this section are stated and proved in [23].

Definition 2.1. *Let $T : L^p(G) \to L^q(G)$ be a bounded linear transformation where $1 \leq p, q \leq \infty$. T is said to be a multiplier of $(L^p(G), L^q(G))$ if T commutes with every translation operator.*

We denote by $M(L^p(G))$ the space of all multipliers on $L^p(G)$, $\parallel T \parallel_p$ the operator norm of each $T \in M(L^p(G))$, $M(L^p(G), L^q(G))$ the space of all multipliers of $(L^p(G), L^q(G))$, and $\parallel T \parallel_{p,q}$ the operator norm of each $T \in M(L^p(G), L^q(G))$.

Theorem 2.2. *Let T be a linear operator on $L^1(G)$. Then the following are equivalent:*

(i) $T \in M(L^1(G))$;

(ii) $T(f * g) = Tf * g$ for each $f, g \in L^1(G)$;

(iii) There exists a unique measure $\mu \in M(G)$ such that $T = T_\mu : f \to \mu * f, f \in L^1(G)$.

Moreover, the correspondence between T and μ defines an isometric algebra isomorphism from $M(L^1(G))$ onto $M(G)$.

Theorem 2.3. *Let $T : L^1(G) \to L^p(G)$ be a linear operator, $1 < p \leq \infty$. then the following are equivalent:*

(i) $T \in M(L^1(G), L^p(G))$;

(ii) There exists a unique function $g \in L^p(G)$ such that $T = T_g : f \to g * f, f \in L^1(G)$.

Moreover, the correspondence between T and g defines an isometric isomorphism from $M(L^1(G), L^p(G))$ onto $L^p(G)$.

Theorem 2.4. *Let T be a linear operator on $L^2(G)$. Then the following are equivalent:*

(i) $T \in M(L^2(G))$;

(ii) There exists a unique function $\varphi \in L^\infty(\widehat{G})$ such that $\widehat{T(f)} = \varphi \widehat{f}$ for each $f \in L^2(G)$.

Moreover, the correspondence between T and φ defines an isometric algebra isomorphism from $M(L^2(G))$ onto $L^\infty(\widehat{G})$.

Corollary 2.5. *If $T \in M(L^p(G))$, where $1 < p < \infty$, then there exists a unique $\varphi \in L^\infty(\widehat{G})$ such that $\widehat{T(f)} = \varphi \widehat{f}$ for each $f \in L^2(G) \cap L^p(G)$. Further, we have $\| \varphi \|_{L^\infty(\widehat{G})} = \| T \|_2 \leq \| T \|_p$.*

Definition 2.6. *Let $p > 1$ and p' such that $\frac{1}{p} + \frac{1}{p'} = 1$. We define the Figà-*

Talamanca space $A^p(G)$ as follows:

$$A^p(G) =: \{\sum_{i=1}^{\infty} f_i * g_i : f_i, g_i \in \mathcal{C}_c(G) \ \textit{and} \ \sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_{p'} < \infty\},$$

endowed with the norm

$$|||f||| =: \inf\{\sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_{p'} : f_i, g_i \in \mathcal{C}_c(G) \ \textit{and} \ f = \sum_{i=1}^{\infty} f_i * g_i\}.$$

Proposition 2.7. $(A^p(G), |||f|||)$ *is a Banach space.*

Theorem 2.8. *There exists an isometric linear isomorphism of $M(L^p(G))$ into $(A^p(G))^*$, the Banach space of continuous linear functionals on $A^p(G)$.*

There is a similar characterization of $M(L^p(G), L^q(G))$ in term of Figà-Talamanca spaces ([12, [13], and [23]).

2.2 Beurling weighted spaces

Definition 2.9. *A measurable function ω on G is said to be a Beurling weight if it has the following properties:*

(i) $\omega(x) \geq 1$ for every $(x \in G)$;

(ii) $\omega(x + y) \leq \omega(x)\omega(y)$ for every $(x, y \in G)$;

(iii) ω is locally bounded, i.e., ω is bounded on every compact subset of G.

For our study we restrict ourselves to the abelian case, but the definition above may be stated for any locally compact group. There are also some more generalized definitions of weight functions ([15], [30], and [17]).

Example 2.10. $\omega(x) = (1+ \mid x \mid)^{\alpha}$, *where $\alpha > 0$, is a Beurling weight.*

Example 2.11. $\omega(x) = e^{\lambda|x|^\alpha}$, *where* $\alpha, \lambda > 0$, *is a Beurling weight.*

Example 2.12. *If* (a_n) *is a sequence of positive numbers, satisfying the conditions*

$$\sum_{n\geq 1}(a_n)^{\frac{1}{n}} < \infty \quad and \quad a_{m+n}(m+n)! \leq a_m m! a_n n!,$$

then $\omega(x) = \sum_{n\geq 1} a_n \mid x \mid^n$*is a Beurling weight.*

These examples are stated in [30].

Remark 2.13. *If* ω_1 *and* ω_2 *are two Beurling weights, then so is* $\omega_1\omega_2$.

In the following notation, all functions considered are supposed to be measurables. For $1 \leq p < \infty$, we denote

$$L^p_\omega(G) =: \{f : \|f\|_{p,\omega} =: (\textstyle\int_G \mid f \mid^p \omega dx)^{1/p} < \infty\},$$

$$L^\infty_\omega(G) =: \{f : f\omega \in L^\infty(G)\},$$

and $M_\omega(G) =: \{\mu : \mu$ is a bounded measure and $\|\mu\|_\omega =: \int \omega|\mu| < \infty\}$.

From the definition of ω, we can deduce easily that ωdx is a positive measure on G. Then all the spaces, considered above, are Banach spaces. Let $f, g \in L^1_\omega(G)$. It is easy to check that

$$\|f * g\|_{1,\omega} \leq \|f\|_{1,\omega}\|g\|_{1,\omega}.$$

Thus, $L^1_\omega(G)$ is a Banach algebra for the convolution product [30].

Definition 2.14. *The spaces* $L^p_\omega(G)$, $1 \leq p \leq \infty$, *are called Beurling weighted spaces.* $L^1_\omega(G)$ *is called a Beurling algebra.*

In the following proposition I summarize some properties of Beurling weighted spaces.

Proposition 2.15. *(i)The space $\mathcal{C}_c(G)$ is dense in $L^p_\omega(G)$.*

(ii) $L^1_\omega(G)$ is an algebra without order, i.e., if for a function $f \in L^1_\omega(G)$ we have $f * g = 0$ a.e for each $g \in L^1_\omega(G)$ then $f = 0$ a.e.

(iii) Let $a \in G$. The translation operator $f \to \iota_a f$ is an isomorphism on $L^p_\omega(G)$, and we have $\| \iota_a f \|_{p,\omega} \leq \omega(a) \| f \|_{p,\omega}$.

(iv) If $f \in L^p_\omega(G)$ then the function $x \to \iota_x f$ is continuous on G.

Statements (i), (ii), and (iii) are stated and proved in [30]. I prove statement (iv), since I could not find any reference for this statement.

Proof of Proposition 2.15(iv). Let $\varepsilon > 0$ and $g \in L^p_\omega(G)$. We claim that there exists a compact neighborhood V of e such that $\| \iota_y g - g \|_{p,\omega} \leq \varepsilon$ for all $y \in V$.

First, let us show our claim for $g \in \mathcal{C}_c(G)$. Let $K_1 = \operatorname{supp} g$, let K_2 be a compact neighborhood of e, and set

$$K = K_1 \cup K_2 \cup (K_1 + K_2) \quad \text{and} \quad A = \sup_{x \in K} \omega(x).$$

For all $y \in K_2$, we have

$$\| \iota_y g - g \|^p_{p,\omega} = \int_K | \iota_y g(x) - g(x) |^p \omega(x) dx \leq A \int_K | \iota_y g(x) - g(x) |^p dx.$$

Since g is uniformly continuous, there exists a neighborhood V of e, which we may assume to be contained in K_2, such that

$$| g(x-y) - g(x) |^p < \frac{\varepsilon^p}{A | K |} \quad \text{for all} \quad y \in V,$$

where $| K |$ is the measure of K. Therefore, for $y \in V$, we have

$$\| \iota_y g - g \|^p_{p,\omega} \leq A \int_K | \iota_y g(x) - g(x) |^p dx \leq \frac{A\varepsilon^p | K |}{A | K |} = \varepsilon^p,$$

and this shows our claim for $g \in \mathcal{C}_c(G)$.

Now let us show the claim for $g \in L^p_\omega(G)$. Let K be a compact neighborhood of e. Since $\mathcal{C}_c(G)$ is dense in $L^p_\omega(G)$, there exists $f \in \mathcal{C}_c(G)$ such that

$$\| g - f \|_{p,\omega} < \inf(\frac{\varepsilon}{3}, \frac{\varepsilon}{3A}), \quad \text{where} \quad A = \sup_{x \in K} \omega(x).$$

By the first step, there exists a compact neighborhood V of e, which we may assume to be contained in K, such that

$$\| \iota_y f - f \|_{p,\omega} \leq \frac{\varepsilon}{3} \quad \text{for all} \quad y \in V.$$

Therefore, for all $y \in V$, we have

$$\begin{aligned} \| \iota_y g - g \|_{p,\omega} &\leq \| \iota_y g - \iota_y f \|_{p,\omega} + \| \iota_y f - f \|_{p,\omega} + \| f - g \|_{p,\omega} \\ &\leq \omega(y) \| f - g \|_{p,\omega} + \frac{2\varepsilon}{3} \leq A \frac{\varepsilon}{3A} + \frac{2\varepsilon}{3} = \varepsilon. \end{aligned}$$

This shows our claim which, in other words, means the continuity of the function $x \to \iota_x f$ at e. This is clearly sufficient to deduce the continuity of the function $x \to \iota_x f$ on the group G. $\square$

We define the space

$$\mathcal{F}_\omega(\widehat{G}) =: \{\widehat{f} : f \in L^1_\omega(G)\},$$

and endow it with the norm $\|\widehat{f}\|_{\mathcal{F}_\omega(\widehat{G})} = \|f\|_{1,\omega}$. Since $L^1(G)$ is a commutative Banach algebra for the convolution, $\mathcal{F}_\omega(\widehat{G})$ is a commutative Banach algebra for pointwise multiplication. $\mathcal{F}_\omega(\widehat{G})$ is a subalgebra of $C_0(G)$. If $\omega = 1$ then $\mathcal{F}_\omega(\widehat{G}) = \mathcal{F}(\widehat{G})$, the so called Fourier algebra of the group G ([9], [10], [25], [26], and [30]).

Definition 2.16. *Let ω be a Beurling weight on G.*

(i) We say that Wiener's approximation theorem holds for $\mathcal{F}_\omega(\hat{G})$ if, for all $f \in L^1_\omega(G)$, the linear combination of translates of f are dense in $L^1_\omega(G)$ if and only if $|\widehat{f}(\gamma)| > 0$, for all $\gamma \in \widehat{G}$.

(ii) We say that $\mathcal{F}_\omega(\widehat{G})$ is a Wiener algebra if the continuous compactly supported functions, contained in $\mathcal{F}_\omega(\widehat{G})$, are dense in $\mathcal{F}_\omega(\widehat{G})$.

The following theorem is fundamental for commutative Beurling algebras.

Theorem 2.17. *([9], 1956) Let ω be a Beurling weight on a locally compact abelian group G. Then $\mathcal{F}_\omega(\widehat{G})$ is a Wiener algebra and Wiener's approximation theorem holds for $\mathcal{F}_\omega(\widehat{G})$ if and only if ω satisfies the condition,*

$$\sum_{n\geq 1} \frac{log\ \omega(nt)}{n^2} < \infty, \ \textit{for all } t \in G.$$

The condition of Theorem 2.17 is called the Beurling-Domar condition, or the non-quasi-analyticity condition.

A. Beurling proved Theorem 2.17 in the real case ([4], 1932). In 1956, Y. Domar proved the generalization to locally compact abelian groups ([1], [9], [25], [26] and [30]).

If ω is a Beurling weight on an abelian locally compact group G, then ωdx is a positive Radon measure on G that can be seen as a generalization of dx, the Haar measure on G. Naturally, one wants to check if a certain property P of the measure dx holds for the measure ωdx, or under what condition the property P holds for the measure ωdx. Under the Beurling-Domar condition, Theorem 2.17 shows spectral synthesis similarities between $\mathcal{F}_\omega(\widehat{G})$ and $\mathcal{F}(\widehat{G})$. One of the properties that is not in general shared by dx with ωdx is that a translation operator is an isometry on $L^p(G)$, while it is not in general an isometry on $L^p_\omega(G)$, see Proposition 2.15(iii). This last fact is closely related to multiplier problems.

2.3 Multipliers of $L^1_\omega(G)$

We denote by $M(L^1_\omega(G))$ the space of all multipliers on $L^1_\omega(G)$ and $\| T \|_{1,\omega}$ the operator norm of each $T \in M(L^1_\omega(G))$. If $\mu \in M_\omega(G)$ we denote $\| \mu \|_\omega = \int \omega \mid \mu \mid$, and if $\omega = 1$ we denote $\| \mu \| = \int \mid \mu \mid$, see appendix A.11 for the definition of $\mid \mu \mid$.

Theorem 2.18. *Let T be a linear mapping on $L^1_\omega(G)$. Then the following are equivalent:*

(i) $T \in M(L^1_\omega(G))$;

(ii) $T(f * g) = Tf * g = f * Tg$ for each $f, g \in L^1_\omega(G)$.

Proof. The proof is based on the fact that $L^1_\omega(G)$ is an algebra without order, see Proposition 2.15(ii).

(ii) $\Rightarrow$ (i). Let $f, g, h \in L^1_\omega(G)$ and let a and b be complex numbers. We have

$$f * T(ag + bh) = Tf * (ag + bh) = f * (aTg + bTh).$$

Since f is arbitrary and by using Proposition 2.15(ii), we deduce that

$$T(ag + bh) = aTg + bTh.$$

Let $f, g, h \in L^1_\omega(G)$ and let (g_n) be a sequence in $L^1_\omega(G)$ such that

$$\lim \| g_n - g \|_{1,\omega} = 0 \quad \text{and} \quad \lim \| Tg_n - h \|_{1,\omega} = 0.$$

We have

$$\begin{aligned}\| f * h - f * Tg \|_{1,\omega} &\leq \| f * h - f * Tg_n \|_{1,\omega} + \| f * Tg_n - f * Tg \|_{1,\omega} \\ &\leq \| f \|_{1,\omega} \| h - Tg_n \|_{1,\omega} + \| Tf * g_n - Tf * g \|_{1,\omega} \\ &\leq \| f \|_{1,\omega} \| h - Tg_n \|_{1,\omega} + \| Tf \|_{1,\omega} \| g_n - g \|_{1,\omega} .\end{aligned}$$

If we let n tend to infinity, we obtain $f * (h - Tg) = 0$. Since f is an arbitrary function and by using Proposition 2.15(ii), we obtain $Tg = h$. Finally the closed graph theorem shows that T is continuous, see [23].

It remains to show that T commutes with translations. Let $a \in G$ and let $f, g \in L^1_\omega(G)$. Then

$$T\iota_a f * g = T(\iota_a f * g) = T(f * \iota_a g) = Tf * \iota_a g = \iota_a Tf * g.$$

By using Proposition 2.15(ii) another time, we obtain $T\iota_a = \iota_a T$.

(i) $\Rightarrow$ (ii). Let $\phi \in L^\infty_\omega(G)$. The mapping $f \to \int Tf\phi$ is a continuous linear form on $L^1_\omega(G)$. Thus, there exists a function $\psi \in L^\infty_\omega(G)$ such that

$$\int Tf\phi = \int f\psi\omega, \text{ for all } f \in L^1_\omega(G).$$

Let $f, g \in L^1_\omega(G)$, $\phi \in L^\infty_\omega(G)$, and let ψ be defined as above. Then

$$\begin{aligned}\int [Tf * g](t)\phi(t)dt &= \int\int Tf(t-s)g(s)ds\phi(t)dt = \int\int \iota_s Tf(t)g(s)ds\phi(t)dt \\ &= \int\int T\iota_s f(t)\phi(t)dtg(s)ds = \int\int \iota_s f(t)\psi(t)\omega(t)dtg(s)ds \\ = \int\int \iota_s f(t)g(s)ds\psi(t)\omega(t)dt &= \int [f * g](t)\psi(t)\omega(t)dt = \int T[f * g](t)\phi(t)dt.\end{aligned}$$

Since ϕ is arbitrary, we conclude that $T(f * g) = Tf * g$. Finally, by commutativity, we obtain

$$T(f * g) = Tf * g = f * Tg.$$

$\square$

Theorem 2.19. *Assume that the weight* ω *is continuous. Let* T *be a bounded linear operator on* $L^1_\omega(G)$*. Then*

(i) $T \in M(L^1_\omega(G))$ *if and only if there exists a unique measure* μ *such that* $T = T_\mu : f \to \mu * f, f \in L^1_\omega(G)$.

(ii) $\omega(e) \parallel \mu \parallel_\omega = \parallel T_\mu \parallel$.

(iii) $M(L^1_\omega(G))$ *and* $M_\omega(G)$ *are topologically and algebraically identified.*

Without assuming that the group G is abelian or that ω is continuous, G. Gaudry proved a theorem similar to Theorem 2.19, [15]. I propose a new proof of Gaudry's result. I shall use the following lemma.

Lemma 2.20. *Let* $(f_n)_{n>0}$ *be a bounded sequence in* $L^1(G)$ *with the following properties:*

(i) If K *is an arbitrary, but fixed, compact neighborhood of* e*, then*

$$\lim \int_{G \setminus K} | f_n(x) | \, dx = 0.$$

(ii) $\lim \int | f_n(x) | \, dx = 1$.

Then $(f_n)_{n>0}$ *is an approximate identity in* $L^1(G)$*.*

For a proof of this lemma I refer to [16].

Corollary 2.21. *For each compact neighborhood* K *of* e*, there exists an approximate identity* $(f_n)_{n>0}$ *in* $L^1(G)$ *with the following properties:*

(i) $\int | f_n(x) | \, dx = 1$.

(ii) For each $n > 0$*, the function* f_n *is supported in* K*.*

Proof. Let $(K_n)_{n>0}$ be a sequence of compact neighborhoods of e, such that

(i)' For each $n > 0$, $K_n \subset K$.

(ii)' For each neighborhood E of e there exits $n > 0$ such that $K_n \subset E$.

Consider the sequence $(f_n = \frac{\mathbf{1_{K_n}}}{|K_n|})_{n>0}$. This sequence is bounded in $L^1(G)$ and satisfies conditions (i) and (ii) of Lemma 2.20. Thus, it is an approximate identity in $L^1(G)$. Since the sequence $(f_n)_{n>0}$ also satisfies the condition of Corollary 2.21, the proof is complete. □

Lemma 2.22. *Let $T \in M(L^1_\omega(G))$. Then there exists a unique bounded measure μ such that $T = T_\mu : f \to \mu * f, f \in L^1_\omega(G)$.*

Proof. Fix K, a compact neighborhood of e. Let $(f_n)_{n>0}$ be an approximate identity in $L^1(G)$ satisfying the condition of Corollary 2.21 for the compact set K. Let $f \in L^1(G)$. By Theorem 2.18 we have $T(f_n) * f = f_n * T(f)$. Let $\varepsilon > 0$. Since $(f_n)_{n>0}$ is an approximate identity in $L^1(G)$, there exists an $n > 0$ such that

$$\| T(f_n) * f - T(f) \|_1 = \| f_n * T(f) - T(f) \|_1 < \varepsilon.$$

Then

$$\begin{aligned} \| T(f) \|_1 &\leq \varepsilon + \| T(f_n) * f \|_1 \leq \varepsilon + \| T(f_n) \|_{1,\omega} \| f \|_1 \\ &\leq \varepsilon + \| T \|_{1,\omega} \| f_n \|_{1,\omega} \| f \|_1 . \end{aligned}$$

Let $M = \sup_{x\in K} \omega(x)$. Since K is compact and ω is locally bounded, we see that M is a finite real number. By using (i) of Corollary 2.21, we obtain

$$\| f_n \|_{1,\omega} = \int | f_n(x) | \omega(x)dx \leq M \int | f_n(x) | dx = M.$$

Therefore,

$$\| T(f) \|_1 \leq \varepsilon + M \| T \|_{1,\omega} \| f \|_1 .$$

Since ε is arbitrary, we have

$$\| T(f) \|_1 \leq M \| T \|_{1,\omega} \| f \|_1 .$$

Thus, T is continuous on $L^1_\omega(G)$ considered with the norm of $L^1(G)$. However, $L^1_\omega(G)$ is dense as a subspace of $L^1(G)$. Hence, T can be extended to a multipliers $\overline{T}$ on $L^1(G)$. By Theorem 2.1, there exists a bounded measure μ such that

$$\overline{T} f = \mu * f, f \in L^1(G),$$

and, hence,

$$T f = \mu * f, f \in L^1_\omega(G).$$

The uniqueness of μ is elementary. $\square$

We are now prepared to prove Theorem 2.19.

Proof of Theorem 2.19. Without lost of generality, we may suppose that $\omega(e) = 1$. Let $\varepsilon > 0$. Since ω is continuous at e, there exists a compact neighborhood of e such that:

$$\sup_{x \in K} \omega(x) < 1 + \varepsilon. \tag{2.1}$$

Let $(f_n)_{n>0}$ be an approximate identity in $L^1(G)$ satisfying the condition of Corollary 2.21. Since $\int \mid f_n(x) \mid d(x) = 1$, hence, by (2.1), we have

$$\| f_n \|_{1,\omega} = \int \mid f_n(x) \mid \omega(x) dx \leq 1 + \varepsilon. \tag{2.2}$$

If $T \in M(L^1_\omega(G))$, then, by Lemma 2.22, there exists a bounded measure μ such that $T = T_\mu : f \to \mu * f$. Consider the sequence $\mu_n = (\mu * f_n)\omega$. Then

$$\| \mu_n \| = \| \mu * f_n \|_{1,\omega} = \| T_\mu(f_n) \|_{1,\omega} \leq \| T_\mu \|_{1,\omega} \| f_n \|_{1,\omega},$$

and, by using (2.2), we obtain

$$\| \mu_n \| \leq (1+\varepsilon) \| T_\mu \|_{1,\omega} . \tag{2.3}$$

The inequality (2.3) implies that $(\mu_n)_{n>0}$, as a sequence of bounded measures, is bounded. Therefore, it has a subsequence $(\mu_{n_k})_{k>0}$ weakly convergent to a bounded measure μ_0. This means:

$$\lim_k \int f d((\mu * f_{n_k})\omega - \mu_0) = 0 \quad \text{for each } f \in C_0(G),$$

i.e.,

$$\lim_k \int f\omega d(\mu * f_{n_k} - \frac{\mu_0}{\omega}) = 0 \quad \text{for each } f \in C_0(G).$$

Since ω is continuous, we deduce that the sequence $(\frac{\mu_{n_k}}{\omega})_{k>0}$ is weakly convergent to $\frac{\mu_0}{\omega}$. Hence, the measure $\frac{\mu_0}{\omega}$ is bounded. However, $(f_{n_k})_k$ is an approximate identity in $L^1(G)$, as that $(\mu * f_{n_k})_k$ converges weakly to μ, see appendix A.10. Therefore, $\mu = \frac{\mu_0}{\omega}$ and the measure $\mu\omega$ is bounded. Hence, $\mu \in M_\omega(G)$. Now, by using (2.3), we obtain

$$\| \mu \|_\omega = \| \mu_0 \| \leq (1+\varepsilon) \| T_\mu \|_{1,\omega} .$$

Since ε is arbitrary, we have

$$\| \mu \|_\omega \leq \| T_\mu \|_{1,\omega} . \tag{2.4}$$

Conversely, let $\mu \in M_\omega(G)$ and $f \in L^1_\omega(G)$. Then

$$\| \mu * f \|_{1,\omega} = \int \left| \int f(t-s)\mu(s) \right| \omega(t)dt \leq \int\int | f(t-s) || \mu | (s)\omega(t)dt$$

$$\leq \int\int | f(t) || \mu | (s)\omega(t+s)dt \leq \int\int | f(t) || \mu | (s)\omega(t)\omega(s)dt = \| f \|_{1,\omega} \| \mu \|_\omega .$$

Thus, $T_\mu \in M(L^1_\omega(G))$ and $\| T_\mu \|_{1,\omega} \leq \| \mu \|_\omega$. Finally, using (2.4), we obtain

$$\| T_\mu \|_{1,\omega} = \| \mu \|_\omega .$$

This finishes the proof of Theorem 2.19. $\square$

2.4 Multipliers of $(L^1_\omega(G), L^p_\omega(G))$, $p > 1$

Definition 2.23. *A bounded operator $T : L^1_\omega(G) \to L^p_\omega(G)$ is said to be a multiplier of $(L^1_\omega(G), L^p_\omega(G))$ if T commutes with every translation operator.*

We denote by $M(L^1_\omega(G), L^p_\omega(G))$ the space of all multipliers of $(L^1_\omega(G), L^p_\omega(G))$ and by $\| T \|_{1,p,\omega}$ the operator norm of each $T \in M(L^1_\omega(G), L^p_\omega(G))$. If $f \in L^p_\omega(G)$ and $f \in L^{p'}_\omega(G)$ where $\frac{1}{p} + \frac{1}{p'} = 1$, we denote $< f, g >_\omega = \int f\overline{g}\omega dx$, and if $\omega = 1$ we denote $< f, g > = < f, g >_\omega$.

The following theorem is the first new result I prove in this chapter.

Theorem 2.24. *Let $T : L^1_\omega(G) \to L^p_\omega(G)$ be a bounded linear transformation, where $p > 1$. Then*

(i) $T \in M(L^1_\omega(G), L^p_\omega(G))$ if and only if there exists a unique function $g \in L^p_\omega(G)$ such that $T = T_g : f \to g * f$, $f \in L^1_\omega(G)$.

(ii) There exists a constant $c \geq 1$ dependent only on the weight function ω,

such that

$$\| T_g \|_{1,p,\omega} \leq \| g \|_{p,\omega} \leq c \| T_g \|_{1,p,\omega} .$$

(iii) $M(L^1_\omega(G), L^p_\omega(G))$ *and* $L^p_\omega(G)$ *are topologically and algebraically identified by the mapping of part (i).*

The proof is based on three new lemmas.

Lemma 2.25. *Let* $T \in M(L^1_\omega(G), L^p_\omega(G))$. *Then*

$$T(f * g) = Tf * g = f * Tg \quad \textit{for each} \quad f, g \in L^1_\omega(G).$$

Proof. Let p' be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Denote for $f \in L^p_\omega(G)$ and $g \in L^{p'}_\omega(G)$

$$< f, g >_\omega = \int f(t)\overline{g(t)}\omega(t)dt.$$

Let $T^* : L^{p'}_\omega(G) \to L^\infty_\omega(G)$ be the adjoint operator of the operator T. Let $f, g \in L^1_\omega(G)$ and let $\varphi \in L^{p'}_\omega(G)$. Then

$$\begin{aligned} < Tf * g, \varphi >_\omega &= \int\int \iota_s Tf(t)g(s)ds\overline{\varphi(t)}\omega(t)dt \\ = \int\int T\iota_s f(t)g(s)ds\overline{\varphi(t)}\omega(t)dt &= \int\int T\iota_s f(t)\overline{\varphi(t)}\omega(t)dtg(s)ds \\ = \int < T\iota_s f, \varphi >_\omega g(s)ds &= \int < \iota_s f, T^*\varphi >_\omega g(s)ds \\ = \int\int \iota_s f(t)\overline{T^*\varphi(t)}\omega(t)dtg(s)ds &= \int\int \iota_s f(t)g(s)ds\overline{T^*\varphi(t)}\omega(t)dt \\ &= < T(f * g), \varphi >_\omega . \end{aligned}$$

Since φ is arbitrary, $T(f * g) = Tf * g$ and by commutativity we achieve the proof of Lemma 2.25. □

For the rest of this section, let K be a compact neighborhood of e and set

$$c = \sup_{x \in K} \omega(x). \tag{2.5}$$

Lemma 2.26. *For each $g \in L^p_\omega(G)$ and each $\varepsilon > 0$ there exists a positive function $h \in \mathcal{C}_c(G)$ satisfying the following conditions:*

$$\| h \|_1 = 1, \quad \| h \|_{1,\omega} \leq c, \quad \textit{and} \quad \| g * h - g \|_{p,\omega} \leq \varepsilon.$$

Proof. Let $g \in L^p_\omega(G)$ and let $\varepsilon > 0$. By Property 2.15(iv) the function $y \to \iota_y g$ is continuous at e. Then there exists a neighborhood V, which we may assume to be contained in K, such that

$$\| \iota_y g - g \|_{p,\omega} \leq \varepsilon \ \text{ for all } \ y \in V.$$

Consider a positive function $h \in \mathcal{C}_c(G)$, such that

$$\text{supp } h \subset V \ \text{ and } \ \int h(y)dy = 1.$$

By using (2.5) and since $V \subset K$, we have $\int h(y)\omega(y)dy \leq c$.

We also have

$$| (g * h)(x) - g(x) | \leq \int | \iota_y g(x) - g(x) | \, h(y)dy.$$

By Hölder's inequality with respect to the measure $h(y)dy$, we obtain

$$\begin{aligned} | (g * h)(x) - g(x) | &\leq \Big(\int | \iota_y g(x) - g(x) |^p \, h(y)dy\Big)^{\frac{1}{p}} \Big(\int h(y)dy\Big)^{\frac{1}{p'}} \\ &\leq \Big(\int | \iota_y g(x) - g(x) |^p \, h(y)dy\Big)^{\frac{1}{p}}, \end{aligned}$$

where p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$. Therefore,

$$\| g * h - g \|^p_{p,\omega} = \int | g * h(x) - g(x) |^p \, \omega(x)dx$$

$$\leq \int \int | \iota_y g(x) - g(x) |^p \, h(y)dy\omega(x)dx \leq \int \| \iota_y g - g \|^p_{p,\omega} \, h(y)dy = \varepsilon^p.$$

This completes the proof of Lemma 2.26. □

Lemma 2.27. *Let $T \in M(L^1_\omega(G), L^p_\omega(G))$. Then there exists a unique function $g \in L^p(G)$ such that $T = T_g : f \to g * f, f \in L^1_\omega(G)$.*

Proof. We claim that T is continuous if $L^1_\omega(G)$ is endowed with the norm of $L^1(G)$ and $L^p_\omega(G)$ is endowed with the norm of $L^p(G)$.

Let $f \in L^1_\omega(G)$. Then $Tf \in L^p_\omega(G) \subset L^pG)$. For $\varepsilon > 0$, using Lemma 2.26 for $L^p(G)$, there exists a positive function $\varphi \in \mathcal{C}_c(G)$, supported in the compact set K, such that

$$\int \varphi(t)dt = 1 \quad \text{and} \quad \| \varphi * Tf - Tf \|_p < \varepsilon,$$

hence,

$$\| Tf \|_p < \varepsilon + \| \varphi * Tf \|_p = \varepsilon + \| T\varphi * f \|_p \leq \varepsilon + \| T\varphi \|_p \| f \|_1$$

$$\leq \varepsilon + \| T\varphi \|_{p,\omega} \| f \|_1 \leq \varepsilon + \| T \|_{1,p,\omega} \| \varphi \|_{1,\omega} \| f \|_1 .$$

However,

$$\| \varphi \|_{1,\omega} = \int \varphi(t)\omega(t)dt \leq c.$$

Then

$$\| Tf \|_p \leq \varepsilon + \| T \|_{1,p,\omega} \, c \, \| f \|_1 .$$

If we let ε tend to zero, we obtain

$$\| Tf \|_p \leq \| T \|_{1,p,\omega} \, c \, \| f \|_1$$

and this shows the claim.

Since $L^1_\omega(G)$ is dense as a subspace of $L^1(G)$, T can be extended to a multiplier $\overline{T} \in M(L^1(G), L^p(G))$. By Theorem 2.3, there exists a function $g \in L^p(G)$ such that

$$\overline{T}f = g * f, f \in L^1(G),$$

and hence,

$$Tf = g * f, f \in L^1_\omega(G)$$

The uniqueness of g is elementary. $\square$

Now we are prepared to prove Theorem 2.24.

Proof of Theorem 2.24. Let $g \in L^p_\omega(G)$ and $f \in L^1_\omega(G)$. Then by using Hölder's inequality with respect to the measure $\mid f(y) \mid dy$, we obtain

$$\begin{aligned}
\| g * f \|_{p,\omega}^p = \int \mid g * f \mid^p \omega(x)dx &= \int \mid \int g(x-y)f(y)dy \mid^p \omega(x)dx \\
&\leq \int [\int \mid g(x-y) \mid^p \mid f(y) \mid dy][\int \mid f(y) \mid dy]^{\frac{p}{p'}} \omega(x)dx \\
&\leq \int (\mid g \mid^p * \mid f \mid)\omega(x)dx[\int \mid f(y) \mid dy]^{\frac{p}{p'}} \\
\leq \| \mid g \mid^p * \mid f \mid \|_{1,\omega} \| f \|_{1,\omega}^{\frac{p}{p'}} \leq \| g \|_{p,\omega}^p \| f \|_{1,\omega} \| f \|_{1,\omega}^{\frac{p}{p'}} &= \| g \|_{p,\omega}^p \| f \|_{1,\omega}^p .
\end{aligned}$$

Therefore, $T_g \in M(L^1_\omega(G), L^p_\omega(G))$ and

$$\| T_g \|_{1,p,\omega} \leq \| g \|_{p,\omega} . \tag{2.6}$$

Let $\varepsilon > 0$. By Lemma 2.26, for each $g \in L^p_\omega(G)$, there exists a positive function h such that

$$\| h \|_1 = 1, \quad \| h \|_{1,\omega} \leq c, \quad \text{and} \quad \| g * h - g \|_{p,\omega} \leq \varepsilon.$$

Then

$$\| g \|_{p,\omega} \leq \varepsilon + \| g * h \|_{p,\omega} = \varepsilon + \| T_g(h) \|_{p,\omega}$$

$$\leq \varepsilon + \| T_g \|_{1,p,\omega} \| h \|_{1,\omega} \leq \varepsilon + c \| T_g \|_{1,p,\omega} .$$

Since ε is arbitrary, we obtain

$$\| g \|_{p,\omega} \leq c \| T_g \|_{1,p,\omega} . \tag{2.7}$$

Conversely, let $T \in M(L^1_\omega(G), L^p_\omega(G))$. By Lemma 2.27, there exists a unique function $g \in L^p(G)$ such that $T = T_g$. It suffices to show that, in fact, $g \in L^p_\omega(G)$. Using Lemma 2.26 for $L^p(G)$, it is easy to construct for g a sequence $(h_n)_n$ satisfying, for each integer $n > 0$, the following conditions:

$$\| g * h_n - g \|_p < \frac{1}{n}, \tag{2.8}$$

$$\| h_n \|_1 = 1, \tag{2.9}$$

$$\text{and} \quad \| h_n \|_{1,\omega} \leq c, \tag{2.10}$$

where the inequality (2.10) is obtained by assuming that supp $h_n \subset K$, for each $n > 0$. Then we have

$$\| g * h_n \|_{p,\omega} = \| T_g(h_n) \|_{p,\omega} \leq \| T_g \|_{1,p,\omega} \| h_n \|_{1,\omega} \leq c \| T_g \|_{1,p,\omega} .$$

Thus, the sequence $(g * h_n)_n$ is bounded in $L^p_\omega(G)$. Therefore, it has a subsequence $(g * h_{n_k})_k$, weakly convergent in $L^p_\omega(G)$ to a function $g_0 \in L^p_\omega(G)$. This means:

$$\lim_k < g * h_{n_k}, f >_\omega = < g_0, f >_\omega \quad \text{for all} \quad f \in L^{p'}_\omega(G),$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, i.e.,

$$\lim_k < g * h_{n_k}, f\omega > = < g_0, f\omega > \quad \text{for all} \quad f \in L^{p'}_\omega(G).$$

Since for every $u \in L^{p'}_\omega(G)$ we can write $u = \frac{u}{\omega}\omega$ and $\frac{u}{\omega} \in L^{p'}(G)$, we obtain

$$\lim_k < g * h_{n_k}, u > = < g_0, u > \quad \text{for all} \quad u \in L^{p'}(G).$$

Therefore, the sequence $(g * h_{n_k})_k$ converges weakly in $L^p(G)$ to the function g_0. However, it follows from inequality (2.8) that the sequence $(g * h_{n_k})_k$ converges strongly, and hence, weakly, to the function g. Therefore, $g = g_0$ and thus $g \in L^p_\omega(G)$. This combined with (2.6) and (2.7) completes the proof of Theorem 2.24. $\square$

Remark 2.28. *If ω is continuous at e and $\omega(e) = 1$, then the constant c in Theorem 2.24 can be taken to be 1, and the correspondence between $L^p_\omega(G)$ and $M(L^1_\omega(G), L^p_\omega(G))$ is an isometry.*

2.5 Multipliers of $L^p_\omega(G)$ and of $(L^p_\omega(G), L^p(G))$, $p > 1$

Definition 2.29. *A bounded linear operator $T : L^p_\omega(G) \to L^p_\omega(G)$ (or $T : L^p_\omega(G) \to L^p(G)$) is said to be a multiplier of $L^p_\omega(G)$ (or of $(L^p_\omega(G), L^p(G))$) if T commutes with every translation operator.*

We denote by $M(L^p_\omega(G))$ the space of all multipliers of $L^p_\omega(G)$ and by $M(L^p_\omega(G), L^p(G))$ the space of all multipliers of $(L^p_\omega(G), L^p(G))$.

I prove the following new lemma.

Lemma 2.30. . *Let $T \in M(L^p_\omega(G))$ (or $T \in M(L^p_\omega(G), L^p(G))$). Then*

$$Tf * g = T(f * g) = f * Tg \quad \textit{for all} \quad f, g \in L^1_\omega(G) \cap L^p_\omega(G).$$

Proof. Let $T \in M(L^p_\omega(G))$ and $f, g \in \mathcal{C}_c(G)$. Consider $T^* : L^{p'}_\omega(G) \to L^{p'}_\omega(G)$, the adjoint of T, where $\frac{1}{p} + \frac{1}{p'} = 1$. Let $\varphi \in L^{p'}_\omega(G)$. We have

$$< Tf * g, \varphi >_\omega = \int\int \iota_s Tf(t) g(s) ds \overline{\varphi(t)} \omega(t) dt$$
$$= \int\int T\iota_s f(t) g(s) ds \overline{\varphi(t)} \omega(t) dt = \int\int T\iota_s f(t) \overline{\varphi(t)} \omega(t) dt g(s) ds$$
$$= \int < T\iota_s f, \varphi >_\omega g(s) ds = \int < \iota_s f, T^*\varphi >_\omega g(s) ds$$

$$= \int\int \iota_s f(t)\overline{T^*\varphi(t)}\omega(t)dtg(s)ds = \int\int \iota_s f(t)g(s)ds\overline{T^*\varphi(t)}\omega(t)dt$$

$$=< T(f*g), \varphi >_\omega .$$

We were able to use Fubini's theorem in these equalities because $f, g \in \mathcal{C}_c(G)$. Since φ is arbitrary, we have $T(f*g) = Tf*g$; and, by commutativity, we obtain

$$Tf*g = T(f*g) = f*Tg \quad \text{for all} \quad f, g \in \mathcal{C}_c(G).$$

Now let $f, g \in L^1_\omega(G) \cap L^p_\omega(G)$ and let (f_n) and (g_n) be two sequences in $\mathcal{C}_c(G)$ such that $\lim \| f_n - f \|_{p,\omega} = \lim \| g_n - g \|_{1,\omega} = 0$. We have

$$\| T(f*g) - Tf*g \|_{p,\omega} \leq \| T(f*g) - T(f_n*g_n) \|_{p,\omega} + \| Tf_n*g_n - Tf*g \|_{p,\omega}$$

$$\leq \| T \|_{p,\omega} \| f*g - f_n*g_n \|_{p,\omega} + \| Tf_n*g_n - Tf_n*g \|_{p,\omega} + \| Tf_n*g - Tf*g \|_{p,\omega}$$

$$\leq \| T \|_{p,\omega} [\| f*g - f*g_n \|_{p,\omega} + \| f*g_n - f_n*g_n \|_{p,\omega}]$$

$$+ \| T \|_{p,\omega} \| f_n \|_{p,\omega} \| g_n - g \|_{1,\omega} + \| T \|_{p,\omega} \| f_n - f \|_{p,\omega} \| g \|_{1,\omega}$$

$$\leq \| T \|_{p,\omega} [\| f \|_{p,\omega} \| g - g_n \|_{1,\omega} + \| f - f_n \|_{p,\omega} \| g_n \|_{1,\omega}]$$

$$+ \| T \|_{p,\omega} \| f_n \|_{p,\omega} \| g_n - g \|_{1,\omega} + \| T \|_{p,\omega} \| f_n - f \|_{p,\omega} \| g \|_{1,\omega} .$$

Since $(\| f_n \|_{p,\omega})$ and $(\| g_n \|_{1,\omega})$ are bounded, by letting n tend to infinity we obtain $T(f*g) = Tf*g$; and, by commutativity, we deduce that

$$Tf*g = T(f*g) = f*Tg \quad \text{for all} \quad f, g \in L^1_\omega(G) \cap L^p_\omega(G).$$

The proof for $T \in M(L^p_\omega(G), L^p(G))$ is similar. □

I prove the following new lemma.

Lemma 2.31. *Let μ be a positive measure and let f be a positive measurable function. Then*

$$(\frac{\mu}{\omega} * f) \leq \frac{1}{\omega}(\mu * \omega f).$$

Proof. We have

$$\begin{aligned}(\frac{\mu}{\omega} * f)(x) = \int f(x-y)\frac{1}{\omega(y)}d\mu(y) &= \int f(x-y)\frac{1}{\omega(x-y)}\frac{1}{\omega(x-y)\omega(y)}d\mu(y) \\ &\leq \int f(x-y)\frac{1}{\omega(x-y)}\frac{1}{\omega(x)}d\mu(y) = \frac{1}{\omega}(\mu * \omega f)\end{aligned}$$

and this finishes the proof of Lemma 2.31. □

We say that a Banach space X can be *embedded* into another Banach space Y, if there exists a continuous linear injection from X into Y. The following proposition is a new result.

Proposition 2.32. *(i) The space $M_{\omega^{\frac{1}{p}}}(G)$ can be embedded into the space $M(L^p_\omega(G))$.*

(ii) The space $M(L^p(G))$ can be embedded into the space $M(L^p_\omega(G), L^p(G))$.

(iii) The space $M(L^p_\omega(G))$ can be embedded into the space $M(L^p_\omega(G), L^p(G))$.

Proof. (i) Note that $\omega^{\frac{1}{p}}$ is a Beurling weight and put $\mu_0 = \omega^{\frac{1}{p}}\mu$. Let $f \in L^p_\omega(G)$. Then, by using Lemma 2.31, we obtain

$$\begin{aligned}\| \mu * f \|_{p,\omega} = \| \frac{\mu_0}{\omega^{\frac{1}{p}}} * f \|_{p,\omega} \leq \| \frac{1}{\omega^{\frac{1}{p}}}(| \mu_0 | * | \omega^{\frac{1}{p}} f |) \|_{p,\omega} &= \|| \mu_0 | * | \omega^{\frac{1}{p}} f |\|_p \\ \leq \| \mu_0 \| \| \omega^{\frac{1}{p}} f \|_p = \| \mu\omega^{\frac{1}{p}} \| \| \omega^{\frac{1}{p}} f \|_p &= \| \mu \|_{\omega^{\frac{1}{p}}} \| f \|_{p,\omega} .\end{aligned}$$

Thus, $T_\mu \in M(L^p_\omega(G))$ and $\| T_\mu \|_{p,\omega} \leq \| \mu \|_{\omega^{\frac{1}{p}}}$.

Now, let $\mu \in M_{\omega^{\frac{1}{p}}}(G)$ such that $T_\mu = 0$. Since $\mathcal{C}_c(G) \subset L^p_\omega(G)$, we have

$$T_\mu f = \mu * f = 0 \quad \text{for each} \quad f \in \mathcal{C}_c(G).$$

Thus, $\mu = 0$. We conclude that the correspondence $\mu \to T_\mu$ is a continuous linear injection from $M_{\omega^{\frac{1}{p}}}(G)$ into $M(L^p_\omega(G))$.

(ii) For each $T \in M(L^p(G))$ consider $\psi(T) : L^p_\omega(G) \to L^p(G)$ defined by

$$\psi(T)f = Tf \quad \text{for each} \quad f \in L^p_\omega(G).$$

For each $f \in L^p_\omega(G)$, we have

$$\| \psi(T)f \|_p = \| Tf \|_p \leq \| T \|_{M(L^p(G))} \| Tf \|_p .$$

Then $\| \psi(T) \|_{M(L^p_\omega(G), L^p(G))} \leq \| T \|_{M(L^p(G))}$. Hence, ψ is a continuous linear injection from $M(L^p(G))$ into $M(L^p_\omega(G), L^p(G))$.

Now let $T \in M(L^p(G))$ such that $\psi(T) = 0$. Then

$$\psi(T)f = Tf = 0 \quad \text{for each } f \in L^p_\omega(G).$$

Since $L^p_\omega(G)$ is a dense subspace of $L^p(G)$ and T is continuous on $L^p(G)$, we conclude that $T = 0$. Therefore, ψ realizes an embedding of $M(L^p(G))$ into $M(L^p_\omega(G), L^p(G))$.

(iii) For each $T \in M(L^p_\omega(G))$ consider $\psi(T) : L^p_\omega(G) \to L^p(G)$ defined by

$$\psi(T)f = Tf \quad \text{for each} \quad f \in L^p_\omega(G).$$

Obviously ψ is a linear injection from $M(L^p_\omega(G))$ into $M(L^p_\omega(G), L^p(G))$. Moreover, for each $f \in L^p_\omega(G)$, we have

$$\| \psi(T)f \|_p = \| Tf \|_p \leq \| Tf \|_{p,\omega} \leq \| T \|_{M(L^p_\omega(G))} \| f \|_{p,\omega} .$$

Then

$$\| T \|_{M(L^p_\omega(G), L^p(G))} \leq \| T \|_{M(L^p_\omega(G))} .$$

Therefore, ψ realizes an embedding of $M(L^p_\omega(G))$ into $M(L^p_\omega(G), L^p(G))$. $\square$

This embedding has motivated me to identify $M(L^p_\omega(G), L^p(G))$ with the dual of a new space of Figà-Talamanca type ([12], [13], and [23]), that I shall now define.

Definition 2.33. *Let $p > 1$ and p' such that $\frac{1}{p} + \frac{1}{p'} = 1$. We define the space*

$$A^p_\omega(G) := \{\sum_{i=1}^{\infty} f_i * g_i : f_i, g_i \in \mathcal{C}_c(G) \text{ and } \sum_{i=1}^{\infty} \|f_i\|_{p,\omega} \|g_i\|_{p'} < \infty\},$$

endowed with the norm

$$|||f|||_\omega = \inf\{\sum_{i=1}^{\infty} \|f_i\|_{p,\omega} \|g_i\|_{p'} : f_i, g_i \in \mathcal{C}_c(G) \text{ and } f = \sum_{i=1}^{\infty} f_i * g_i\}.$$

Proposition 2.34. *$(A^p_\omega(G), |||f|||_\omega)$ is a Banach space.*

Proof. Let $(h_n)_{n\geq 1}$ be a Cauchy sequence of $A^p_\omega(G)$. Then $(h_n)_{n\geq 1}$ has a subsequence $(k_n)_{n\geq 1}$ such that $|||k_{n+1} - k_n|||_\omega < 2^{-n}$, for all $n \geq 1$. From the definition of $A^p_\omega(G)$, we can find two sequences $(f_{nj})_{j\geq 1}$ and $(g_{nj})j \geq 1$ such that

(*i*) $k_1 = \sum_{j\geq 1} f_{1j} * g_{1j}$,

(*ii*) $\sum_{j\geq 1} \|f_{1j}\|_{p,\omega} \|g_{1j}\|_q \leq \| k_1 \|_\omega +1$,

(*iii*) $k_{n+1} - k_n = \sum_{j\geq 1} f_{n+1j} * g_{n+1j}$,

(*iv*) $\sum_{j\geq 1} \|f_{n+1j}\|_{p,\omega} \|g_{n+1j}\|_q \leq 2^{-n+1}, n = 1, 2...$.

If we set

$$h = \sum_{j\geq 1} f_{1j} * g_{1j} + \sum_{n\geq 1} \sum_{j\geq 1} f_{n+1j} * g_{n+1j},$$

we obtain

$$\begin{aligned} |||h|||_\omega &\leq \sum_{j\geq 1} \|f_{1j}\|_{p,\omega} \|g_{1j}\|_q + \sum_{n\geq 1} \sum_{j\geq 1} \|f_{n+1j}\|_{p,\omega} \|g_{n+1j}\|_q \\ &\leq |||k_1|||_\omega + 1 + \sum_{n\geq 1} 2^{-n+1} = |||k_1|||_\omega + 3. \end{aligned}$$

Thus, $h \in A^p_\omega(G)$. We have

$$|||h - k_{n+1}|||_\omega = |||h - k_1 - \sum_{m=1}^{n} k_{n+1}|||_\omega = ||| \sum_{m=n+1}^{\infty} \sum_{j\geq 1} f_{m+1j} * g_{m+1j}|||_\omega$$
$$\leq \sum_{m=n+1}^{\infty} \sum_{j\geq 1} \|f_{m+1j}\|_{p,\omega}\|g_{m+1j}\|_q \leq \sum_{m=n+1}^{\infty} 2^{-m+1} = 2^{-n+1}.$$

If we let n tend to infinity, we obtain $\lim |||h - k_{n+1}|||_\omega = 0$, and this shows that $(A^p_\omega(G), |||f|||_\omega)$ is a Banach space. □

From now on, I shall denote by $\| \ T \ \|_{p,\omega}$ the operator norm of each $T \in M(L^p_\omega(G), L^p(G))$. The following lemma is a new result.

Lemma 2.35. *If $T \in M(L^p_\omega(G), L^p(G))$ then there exists a net of functions $(g_\alpha) \subset \mathcal{C}_c(G)$ such that if $T_\alpha = g_\alpha * f$, $f \in L^p_\omega(G)$, then*

(i) $\| \ T_\alpha(f) \ \|_p \leq c \ \| \ T \ \|_{p,\omega} \| \ f \ \|_{p,\omega}$ *for each* $f \in L^p_\omega(G)$ *and for each* α,

(ii) $\lim\limits_\alpha \| \ T_\alpha f - Tf \ \|_p = 0$ *for each* $f \in L^p_\omega(G)$,

where c is a positive constant dependent only on the weight function ω.

To prove this important lemma, I shall follow the same steps used to show its analogous for the classical case in [23].

Proof. We claim that it is sufficient, in order to establish the desired conclusion, to show (i) and

$(ii)'$ $\lim\limits_\alpha < T_\alpha f, g > = < Tf, g >$ for each $f \in L^p_\omega(G)$ and for each $g \in L^{p'}(G)$.

As suppose this were true. Let $a_1, a_2, ..., a_n$ be nonnegative real numbers for which $\sum_{i=1}^n a_i = 1$. Then

$$\| \sum_{i=1}^{n} a_i T_{\alpha_i} f \ \|_p \leq \sum_{i=1}^{n} a_i \ \| \ T_{\alpha_i} f \ \|_p \leq \sum_{i=1}^{n} a_i \ \| \ T_{\alpha_i} \ \|_{p,\omega} \| \ f \ \|_{p,\omega}$$

$$\leq \sum_{i=1}^{n} a_i \parallel T \parallel_{p,\omega} \parallel f \parallel_{p,\omega} = \parallel T \parallel_{p,\omega} \parallel f \parallel_{p,\omega},$$

and

$$\lim_{\alpha} < \sum_{i=1}^{n} a_i T_{\alpha_i} f, g >= \lim_{\alpha} \sum_{i=1}^{n} a_i < T_{\alpha_i} f, g >= 0,$$

for any choice of the a_i and each $f \in L^p_\omega(G)$, $g \in L^{p'}(G)$. Hence, the set of T_α satisfying (i) and (ii)' is convex. The statement (ii) means that (T_α) converges to T in the strong operator topology, while the statement (ii)' means that (T_α) converges to T in the weak operator topology. And since the closure of a convex set in the weak and strong operator topologies are identical, we conclude that there exists a net of functions $(g_\alpha) \subset \mathcal{C}_c(G)$ satisfying (i) and (ii) of Lemma 2.35.

Now let $(u_\beta)_{\beta \geq 1}$ be an approximate identity in $L^1(G)$ such that for each $\beta \geq 1$: $u_\beta \in \mathcal{C}_c(G) * \mathcal{C}_c(G)$, $u_\beta > 0$, $\int u_\beta = 1$, and u_β is zero off of some common compact K_0. Let $(v_\delta)_{\delta \geq 1}$ be an approximate identity in $L^1(\widehat{G})$ such that for each $\delta \geq 1$: $\widehat{v_\delta} \in \mathcal{C}_c(G)$ and $\parallel v_\delta \parallel_1 = 1$. For the existence of approximate identities as $(u_\beta)_{\beta \geq 1}$ or $(v_\delta)_{\delta \geq 1}$ see [16] or [23]. For each $\beta \geq 1$ we have $u_\beta = u^1_\beta * u^2_\beta$ for some $u^1_\beta, u^2_\beta \in \mathcal{C}_c(G)$. By Lemma 2.30, we have $Tu_\beta = Tu^1_\beta * u^2_\beta$. Since $Tu^1_\beta \in L^p(G)$ and $u^2_\beta \in \mathcal{C}_c(G) \subset L^{p'}(G)$, with $\frac{1}{p} + \frac{1}{p'} = 1$, then $Tu_\beta \in \mathcal{C}_0(G)$. Hence, $\widehat{v_\delta} Tu_\beta \in \mathcal{C}_c(G)$ for each β and δ. Ordering the set $\{(\beta, \delta) : \beta \geq 1, \delta \geq 1\}$ lexicographically, we obtain a net $(g_\alpha) \subset \mathcal{C}_c(G)$ upon setting $g_\alpha = \widehat{v_\delta} Tu_\beta$ whenever $\alpha = (\beta, \delta)$. We claim that the net (T_α), defined by $T_\alpha f = g_\alpha * f$ for each $f \in L^p_\omega(G)$, satisfies (i) and (ii)'.

Indeed, for $f, g \in \mathcal{C}_c(G)$, and each α we have

$$|< T_\alpha f, g >| = |< g_\alpha * f, g >| = |< g_\alpha, f * g >|$$

$$= | \int_G g_\alpha(-s)(f * g)(s) ds = | \int_G (\widehat{v_\delta} Tu_\beta)(-s)(f * g)(s) ds |$$

$$=\left| \int_G \int_G \int_{\widehat{G}} \widehat{v_\delta}(\gamma)(-s,-\gamma) T u_\beta(-s) f(s-t) g(t) \widehat{d}\gamma dt ds \right|$$

$$\leq \int_{\widehat{G}} | \widehat{v_\delta}(\gamma) | \{ | \int_G \int_G (s,\gamma) T u_\beta(-s) f(s-t) g(t) dt ds | \} \widehat{d\gamma}$$

$$\leq \int_{\widehat{G}} | \widehat{v_\delta}(\gamma) | \{ | \int_G \int_G T u_\beta(-s)(s-t,\gamma) f(s-t) g(t)(t,\gamma) dt ds | \} \widehat{d\gamma}$$

$$\leq \int_{\widehat{G}} | \widehat{v_\delta}(\gamma) | \{ | \int_G T u_\beta(-s)(\gamma f * \gamma g)(s) ds | \} \widehat{d\gamma}$$

$$\leq \| \widehat{v_\delta} \|_1 \sup_{\gamma\in\widehat{G}} |< T u_\beta, \gamma f * \gamma g >| = \sup_{\gamma\in\widehat{G}} |< T u_\beta, \gamma f * \gamma g >|$$

$$= \sup_{\gamma\in\widehat{G}} |< T(u_\beta * \gamma f), \gamma g >| \leq \sup_{\gamma\in\widehat{G}} \| T(u_\beta * \gamma f) \|_p \| \gamma g \|_{p'}$$

$$\leq \| T \|_{p,\omega} \sup_{\gamma\in\widehat{G}} \| u_\beta * \gamma f \|_{p,\omega} \| \gamma g \|_{p'}$$

$$\leq \| T \|_{p,\omega} \| u_\beta \|_{1,\omega} \sup_{\gamma\in\widehat{G}} \| \gamma f \|_{p,\omega} \| \gamma g \|_{p'} = \| T \|_{p,\omega} \| u_\beta \|_{1,\omega} \| f \|_{p,\omega} \| g \|_{p'} .$$

Since u_β is supported by the compact set K_0, we have

$$\| u_\beta \|_{1,\omega} = \int u_\beta(t)\omega(t) dt \leq c \int u_\beta(t) dt = c,$$

upon putting $c = \sup_{t\in K_0} \omega(t)$. Therefore,

$$|< T_\alpha f, g >| \leq c \| T \|_{p,\omega} \| f \|_{p,\omega} \| g \|_{p'} \quad \text{for each} \quad f, g \in \mathcal{C}_c(G).$$

Then

$$\| T_\alpha f \|_p \leq c \| T \|_{p,\omega} \| f \|_{p,\omega} \quad \text{for each} \quad f \in \mathcal{C}_c(G).$$

By continuity of T_α and density of $\mathcal{C}_c(G)$ in $L^p_\omega(G)$, we obtain

$$\| T_\alpha f \|_p \leq c \| T \|_{p,\omega} \| f \|_{p,\omega} \quad \text{for each} \quad f \in L^p_\omega(G).$$

Let $f, g \in \mathcal{C}_c(G)$ and consider the net of real numbers $X_\alpha =< T_\alpha f, g >$. We have $| X_\alpha | = |< T_\alpha f, g >| \leq c \| T \|_{p,\omega} \| f \|_{p,\omega} \| g \|_{p'}$. Thus, the net $(X_\alpha)_{\alpha\in\mathbb{N}^2}$ has a subnet $(X_\gamma)_{\gamma\in\mathcal{I}}$ convergent to some finite limit l, we recall that $\mathbb{N}^2$ is ordered lexicographically. We claim that $l =< Tf, g >$.

Let $\varepsilon > 0$ and $\gamma = (\beta, \delta) \in \mathcal{I}$. Then

$$\begin{aligned}
&|< Tf, g > - < T_\alpha f, g >| \\
\leq &|< Tf, g > - < u_\beta * Tf, g >| + |< u_\beta * Tf, g > - < T_\alpha f, g >| \\
\leq &|< Tf, g > - < Tf, u_\beta * g >| + |< u_\beta * Tf, g > - < \widehat{v_\delta} T u_\beta * f, g >| \\
\leq &|< Tf, g - u_\beta * g > + |< T u_\beta * f, g > - < \widehat{v_\delta} T u_\beta, f * g >| \\
\leq &\| Tf \|_p \| g - u_\beta * g \|_{p'} + |< (1 - \widehat{v_\delta}) T u_\beta, f * g >| \\
\leq &\| Tf \|_p \| g - u_\beta * g \|_{p'} + \sup_{s \in K} | 1 - \widehat{v_\delta}(-s) | \| T u_\beta \|_p \| f * g \|_{p'},
\end{aligned}$$

where K is the compact support of the function $f * g$. Because the supports of u_β are contained in the compact K_0, it is easily seen that there exists some β_0 such that $\| g - u_{\beta_0} * g \|_q < \frac{\varepsilon}{4}(\| Tf \|_p)^{-1}$. For this β_0, since $\widehat{v_\delta}$ converges uniformly to one on compact subsets of G, there exists a δ_0 such that

$$\sup_{s \in K} | 1 - \widehat{v_{\delta_0}}(-s) | < \frac{\varepsilon}{4}(\| T u_{\beta_0} \|_p \| f * g \|_q)^{-1},$$

and it is obvious that we may choose $(\beta_0, \delta_0) \in \mathcal{I}$. If we put $\gamma_0 = (\beta_0, \delta_0)$, then $|< Tf, g > - X_{\gamma_0} | < \frac{\varepsilon}{2}$. It is always possible to choose γ_0 large enough to have $| X_{\gamma_0} - l | < \frac{\varepsilon}{2}$. Combining these inequalities, we see at once that $|< Tf, g > - l | < \varepsilon$. Consequently, since ε is arbitrary, we conclude that $< Tf, g >= l$. We have shown that if a subnet of $(< T_\alpha f, g >)$ has a limit then this limit is equal to $< Tf, g >$. Therefore, $\lim < T_\alpha f, g >=< Tf, g >$.

Now let $\varepsilon > 0$ and $f \in \mathcal{C}_c(G)$. For each $g \in L^{p'}(G)$ there exists a function $g_0 \in \mathcal{C}_c(G)$ such that

$$\| g - g_0 \|_{p'} < \varepsilon \inf((4c \| T \|_{p,\omega} \| f \|_{p,\omega})^{-1}, (4 \| T \|_{p,\omega} \| f \|_{p,\omega})^{-1})$$

Since $\lim < T_\alpha f, g_0 >=< Tf, g_0 >$, there is an α_0 such that for each α greater than α_0, we have $|< T_\alpha f, g_0 > - < Tf, g_0 >|< \frac{\varepsilon}{2}$. Then

$$|< T_\alpha f, g > - < Tf, g >|\leq|< T_\alpha f, g > - < T_\alpha f, g_0 >|$$

$$+ |< T_\alpha f, g_0 > - < Tf, g_0 >| + |< Tf, g_0 > - < Tf, g >|$$

$$\leq|< T_\alpha f, g - g_0 >| \ +\frac{\varepsilon}{2}+ |< Tf, g_0 - g >|$$

$$\leq\| T_\alpha f \|_p\| g - g_0 \|_{p'} +\frac{\varepsilon}{2}+ \| Tf \|_p\| g - g_0 \|_{p'}$$

$$\leq c \| T \|_{p,\omega}\| f \|_{p,\omega} \varepsilon(4c \| T \|_{p,\omega}\| f \|_{p,\omega})^{-1} + \frac{\varepsilon}{2}$$

$$+ \| T \|_{p,\omega}\| f \|_{p,\omega} \varepsilon(4 \| T \|_{p,\omega}\| f \|_{p,\omega})^{-1} \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon.$$

Hence,

$$\lim < T_\alpha f, g >=< Tf, g > \quad \text{for each} \quad f \in \mathcal{C}_c(G), g \in L^{p'}(G).$$

Finally, let $\varepsilon > 0$ and $g \in L^{p'}(G)$. For each $f \in L^p_\omega(G)$ there exists $f_0 \in \mathcal{C}_c(G)$ such that

$$\| f - f_0 \|_{p,\omega}< \varepsilon \inf((4c \| T \|_{p,\omega}\| g \|_{p'})^{-1}, (4 \| T \|_{p,\omega}\| g \|_{p'})^{-1}).$$

Since $\lim < T_\alpha f_0, g >=< Tf_0, g >$, there is an α_0 such that for each α greater than α_0, we have $|< T_\alpha f_0, g > - < Tf_0, g >|< \frac{\varepsilon}{2}$. Then

$$|< T_\alpha f, g > - < Tf, g >|\leq|< T_\alpha f, g > - < T_\alpha f_0, g >|$$

$$+ |< T_\alpha f_0, g > - < Tf_0, g >| + |< Tf_0, g > - < Tf, g >|$$

$$\leq|< T_\alpha f - T_\alpha f_0, g >| \ +\frac{\varepsilon}{2}+ |< Tf_0 - Tf, g >|$$

$$\leq\| T_\alpha f - T_\alpha f_0 \|_p\| g \|_{p'} +\frac{\varepsilon}{2}+ \| Tf_0 - Tf \|_p\| g \|_{p'}$$

$$\leq c \| T \|_{p,\omega}\| f - f_0 \|_{p,\omega} \varepsilon(4c \| T \|_{p,\omega}\| g \|_{p'})^{-1} + \frac{\varepsilon}{2}$$

$$+ \parallel T \parallel_{p,\omega} \parallel f - f_0 \parallel_{p,\omega} \varepsilon(4 \parallel T \parallel_{p,\omega} \parallel g \parallel_{p'})^{-1} \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon.$$

We conclude that

$$\lim < T_\alpha f, g >=< Tf, g > \quad \text{for each} \quad f \in L^p_\omega(G), g \in L^{p'}(G)$$

which is what we wish to establish. □

Theorem 2.36. *There exists an isometric linear isomorphism of $M(L^p_\omega(G), L^p(G))$ into $(A^p_\omega(G))^*$, the Banach space of continuous linear functionals on $A^p_\omega(G)$.*

Proof. Let $T \in M(L^p_\omega(G), L^p(G))$. If

$$h = \sum_{i=1}^{\infty} f_i * g_i, \quad \text{with} \quad f_i, g_i \in \mathcal{C}_c(G),$$

then set

$$\psi(T)h = \sum_{i=1}^{\infty} < Tf_i, g_i > .$$

Since $Tf_i \in L^p(G))$ and $g_i \in \mathcal{C}_c(G) \subseteq L^{p'}(G))$, $\frac{1}{p} + \frac{1}{p'} = 1$, we conclude that

$$\begin{aligned} | \psi(T)h |=| \sum_{i=1}^{\infty} < Tf_i, g_i >| &\leq \sum_{i=1}^{\infty} \parallel Tf_i \parallel_p \parallel g_i \parallel_{p'} \\ \leq \parallel T \parallel_{p,\omega} \sum_{i=1}^{\infty} \parallel f \parallel_{p,\omega} \parallel g_i \parallel_{p'} &\leq \parallel T \parallel_{p,\omega} ||| h |||_\omega < \infty. \end{aligned} \tag{2.11}$$

It is apparent that $\psi(T)$ is linear. To show that $\psi(T)(h)$ is independent of the representation of h, it suffice to show that $\psi(T)(h) = 0$ whenever $h = 0$. By Lemma 2.35, there exists a net $(g_\alpha) \subset \mathcal{C}_c(G)$ such that:

(*i*) $\parallel T_\alpha(f) \parallel_p \leq c \parallel T \parallel_{p,\omega} \parallel f \parallel_{p,\omega}$ for each $f \in L^p_\omega(G)$ and for each α

(*ii*) $\lim_\alpha \parallel T_\alpha f - Tf \parallel_p = 0$ for each $f \in L^p_\omega(G)$

where $T_\alpha f = g_\alpha * f$ and c is a positive constant dependent only on the weight function ω. Suppose that

$$h = \sum_{i=1}^{\infty} f_i * g_i = 0,\ \ f_i, g_i \in \mathcal{C}_c(G) \ \text{ and } \ \sum_{i=1}^{\infty} \|f_i\|_{p,\omega} \|g_i\|_{p'} < \infty.$$

For each α, we have

$$\sum_{i=1}^{\infty} | < T_\alpha f_i, g_i > | \leq \sum_{i=1}^{\infty} \|T_\alpha f_i\|_p \|g_i\|_{p'} \leq c\|T\|_{p,\omega} \sum_{i=1}^{\infty} \|f_i\|_{p,\omega} \|g_i\|_{p'} < \infty.$$

Hence, the series $\sum_{i=1}^{\infty} < T_\alpha f_i, g_i >$ converges in the supremum norm uniformly with respect to α, and

$$\lim_\alpha \sum_{i=1}^{\infty} < T_\alpha f_i, g_i >= \sum_{i=1}^{\infty} \lim_\alpha < T_\alpha f_i, g_i >= \sum_{i=1}^{\infty} < T f_i, g_i > .$$

We also have

$$\sum_{i=1}^{\infty} < T_\alpha f_i, g_i >= \sum_{i=1}^{\infty} < g_\alpha * f_i, g_i >=< g_\alpha, \sum_{i=1}^{\infty} f_i * g_i > .$$

To justify the last equality, note that the mapping $f \to < g_\alpha, f >$ is a continuous linear form on $\mathcal{C}_0(G)$ and the series $\sum_{i=1}^{\infty} f_i * g_i$ converges in $\mathcal{C}_0(G)$, since

$$\sum_{i=1}^{\infty} \| f_i * g_i \|_\infty \leq \sum_{i=1}^{\infty} \| f_i \|_p \| g_i \|_{p'} \leq \sum_{i=1}^{\infty} \| f_i \|_{p,\omega} \| g_i \|_{p'} < \infty.$$

Consequently, if $h \in A_\omega^p(G)$ is such that $h = 0$ then

$$\sum_{i=1}^{\infty} < T f_i, g_i >= 0.$$

Therefore, $\psi(T)$ is a well defined linear form on $A_\omega^p(G)$. Moreover, by (2.11) we have $\| \psi(T) \| \leq \| T \|_{p,\omega}$. Furthermore, we have

$$\| T \|_{p,\omega} = \sup_{f \in \mathcal{C}_c(G), \|f\|_{p,\omega} \leq 1} \| Tf \|_p$$

$$= \sup_{f\in\mathcal{C}_c(G),\|f\|_{p,\omega}\le 1} \left(\sup_{g\in\mathcal{C}_c(G),\|g\|_q\le 1} |< Tf, g >| \right)$$

$$\le \sup\{|\ \psi(T)(h)\ | : h = f * g, \|\ f\ \|_{p,\omega} \le 1, \|\ g\ \|_{p'} \le 1,\ \text{and}\ f, g \in \mathcal{C}_c(G)\}$$

$$\le \sup\{|\ \psi(T)(h)\ | : \|\|\ h\ \|\|_\omega \le 1\} = \|\ \psi(T)(h)\ \|\ .$$

Hence, $\|\ \psi(T)\ \| = \|\ T\ \|_{p,\omega}$. Therefore,

$$\psi : M(L^p_\omega(G), L^p(G)) \to (A^p_\omega(G))^*$$

is an isometric linear isomorphism. The proof will be complete once we have shown that the mapping ψ is surjective. Let $F \in (A^p_\omega(G))^*$. For each $f \in \mathcal{C}_c(G)$, define $F_f(g) = F(f * g)$, $g \in \mathcal{C}_c(G)$. We have

$$|\ F_f(g)\ | = \|\ F(f * g)\ \| \le \|\ F\ \|\|\|\ f * g\ \|\|_\omega \le \|\ F\ \|\|\ f\ \|_{p,\omega}\|\ g\ \|_{p'}\ .$$

Hence, F_f defines a continuous linear form on $\mathcal{C}_c(G)$ considered as a subspace of $L^{p'}(G)$. Since $\mathcal{C}_c(G)$ is dense in $L^{p'}(G)$ and $(L^{p'}(G))^* = L^p(G)$, there exists a unique function $Sf \in L^p(G)$ such that

$$F_f(g) = F(f * g) =< g, Sf >, \quad \text{for each}\ \ g \in \mathcal{C}_c(G);$$

and we have

$$\|\ Sf\ \|_p \le \|\ F\ \|\|\ f\ \|_{p,\omega} \quad \text{for each}\ \ f \in \mathcal{C}_c(G).$$

Since $\mathcal{C}_c(G)$ is dense in $L^p_\omega(G)$, S can be extended to a continuous linear transformation $T : L^p_\omega(G) \to L^p(G)$, with $\|\ T\ \|_{p,\omega} \le \|\ F\ \|$. Furthermore, for each $s \in G$ and each $f, g \in \mathcal{C}_c(G)$ we have

$$< T\iota_s f, g >= F(\iota_s f * g) = F(f * \iota_s g) =< Tf, \iota_s g >=< \iota_s Tf, g >\ .$$

Since $\mathcal{C}_c(G)$ is dense in $L^{p'}(G)$, we obtain

$$T\iota_s f = \iota_s T f, \quad \text{for each} \quad f \in \mathcal{C}_c(G).$$

Since $\mathcal{C}_c(G)$ is dense in $L^p_\omega(G)$, we obtain

$$T\iota_s f = \iota_s T f, \quad \text{for each} \quad f \in L^p_\omega(G).$$

Thus, $T \in M(L^p_\omega(G), L^p(G))$.

However, if $h \in A^p_\omega(G)$ has the representation $h = \sum_{i=1}^{\infty} f_i * g_i$, then

$$\psi(T)(h) = \sum_{i=1}^{\infty} < Tf_i, g_i >= \sum_{i=1}^{\infty} F(f_i * g_i) = F(h),$$

since the sequence $(\sum_{i=1}^{n} f_i * g_i)_{n\geq 1}$ converges to h in $A^p_\omega(G)$. That is $\psi(T) = F$. Therefore, ψ is surjective and the proof is complete. $\square$

The technique of tensor products is often used to solve multiplier problems for non abelian groups [29]. In [18] A.T.Gürkanli and S.Öztop have used this technique to obtain the following result for a unimodular group and for $1 \leq p \leq 2$:

If every element of $M(L^p_\omega(G), L^p_{\omega^{1-p'}}(G))$ can be approximated in the ultraweak operator topology by an operator of the form $T_\varphi : f \to \varphi * f$, $\varphi \in \mathcal{C}_c(G)$, then $M(L^p_\omega(G), L^p_{\omega^{1-p'}}(G))$ can be identified isometrically with the dual of the Banach space,*

$$'A^p_\omega(G) := \{\sum_{i=1}^{\infty} f_i * g_i : f_i, g_i \in \mathcal{C}_c(G) \ \textit{and} \ \sum_{i=1}^{\infty} \|f_i\|_{p,\omega}\|g_i\|_{p',\omega} < \infty\},$$

endowed with the norm

$$'|||f|||_\omega = \inf\{\sum_{i=1}^{\infty} \|f_i\|_{p,\omega}\|g_i\|_{p',\omega} : f_i, g_i \in \mathcal{C}_c(G) \ \textit{and} \ f = \sum_{i=1}^{\infty} f_i * g_i\}.$$

To avoid confusion, a left prime is added to these newly defined spaces. Let us assume that G is a locally compact abelian group, and $1 < p \leq 2$. Then

$$M(L^p_\omega(G)) \hookrightarrow (A^p_\omega(G))^* \hookrightarrow \ ('A^p_\omega(G))^*.$$

Thus, my result improves the result of A.T.Gürkanli and S.Öztop.

2.6 Summary

A study of multiplier problems for Beurling weighted spaces is presented in this chapter. G. Gaudry solved the multiplier problem for $L^1_\omega(G)$, i.e., he proved that $M(L^1_\omega(G))$ can be identified with the weighted space of bounded measures $M_\omega(G)$ [15]. In this chapter I solved multiplier problems for $(L^1_\omega(G), L^p_\omega(G))$ and for $(L^p_\omega(G), L^p(G))$.

My first new result is Theorem 2.24, where I proved that the spaces $M(L^1_\omega(G), L^p_\omega(G))$ and $L^p_\omega(G)$ can be topologically and algebraically identified. This solves the multiplier problem for $(L^1_\omega(G), L^p_\omega(G))$.

The multiplier problem for $M(L^p_\omega(G))$ is an open and difficult problem [18]. In Proposition 2.32, I proved that $M(L^p_\omega(G))$ can be embedded in the space $M(L^p_\omega(G), L^p(G))$. Motivated by this, I solved the multiplier problem for $(L^p_\omega(G), L^p(G))$. In Definition 2.33, I defined $A^p_\omega(G)$, a new space of Figà-Talamanca type. In Proposition 2.34, I proved that $A^p_\omega(G)$, endowed with a norm defined in Definition 2.33, is a Banach space. In Theorem 2.36, I proved that $M(L^p_\omega(G), L^p(G))$ can be isometrically identified with the dual of the space $A^p_\omega(G)$. This new result improves a result obtained A.T.Gürkanli and S.Öztop [18]. The results obtained in section 2.5 can be summarized as follows:

$$M_{\omega^{\frac{1}{p}}}(G) \hookrightarrow M(L^p_\omega(G)) \hookrightarrow M(L^p_\omega(G), L^p(G)) \simeq (A^p_\omega(G))^*;$$

where the sign $'' \simeq ''$ means an isometric identification, and the sign $'' \hookrightarrow ''$ means an embedding.

Chapter 3

Product-convolution operators

Let G be a locally compact abelian group, $f \in L^1(G)$, and $\varphi \in L^\infty(G)$. We define the convolution and multiplication operators, respectively, as follows:

$$C_f : g \mapsto f * g \text{ and } M_\varphi : g \mapsto \varphi g, \quad g \in L^2(G).$$

If $G = \mathbb{R}$ then, except for trivial cases, the operators C_f and M_φ are never compact on $L^2(\mathbb{R})$. However, the composition of these two operators is, in some cases, compact. A paper by R.C Busby and H.A.Smith [6] gives necessary and sufficient conditions on φ for the compactness of the product-convolution operator $M_\varphi C_f$. In section 3.1, I prove that if φ belongs to the closure of $L^p(G) \cap L^\infty(G)$ in $L^\infty(G)$ and $f \in L^1(G)$, then the product-convolution operator $M_\varphi C_f$ is compact. This can also be deduced from the Busby and Smith results. My proof is based on approximations of compact operators by Hilbert-Schmidt operators and a property of C*-algebras. The proof of R.C Busby and H.A.Smith is based on properties of mixed norm spaces. In section 3.2, I apply the results of the first section to show that some Volterra convolution type integral operators are compact. As a second application, I show that, for any function $f \in L^1(G)$, the convolution operator $C_f : L^2_\omega(G) \to L^2_{\omega^{-1}}(G)$ is compact, where ω is a positive measurable function for which ω^{-1} is bounded and vanishes at infinity. Then I obtain a spectral decomposition of C_f that gives rise to a numerical method to solve a theoretical problem in communication theory [35]. In section 3.3, I prove that the spectral synthesis of some product-convolution operators can be obtained by solving differential equations. This new result is interesting, since it gives the spectral decomposition of some compact operators, which are not necessarily Hilbert-Schmidt operators. I end this section by some relevant examples. Among the consequences of these examples, I obtain special functions as eigenfunctions of product-convolution operators. In section 3.4, I summarize the results of this chapter.

3.1 Compactness of product-convolution operators

Let G be a locally compact abelian group. For each $1 \leq p \leq \infty$ we denote by $\| f \|_p$ the norm of $f \in L^p(G)$. We denote by $\mathcal{L}(L^2(G))$ the space of all bounded linear operators on the Hilbert space $L^2(G)$, and by $\mathcal{LC}(L^2(G))$ the space of all compact operators on $L^2(G)$. We denote by $\| T \|$ the operator norm of each $T \in \mathcal{L}(L^2(G))$. For definitions and certain properties of bounded, compact, and Hilbert-Schmidt operators see appendix B. For $f \in L^1(G)$ and $\varphi \in L^\infty(G)$, we define the convolution and multiplication operators, respectively, as follows:

$$C_f : g \mapsto f * g \quad \text{and} \quad M_\varphi : g \mapsto \varphi g, \quad g \in L^2(G).$$

Theorem 3.1. *Let $f \in L^1(G)$ and $\varphi \in L^\infty(G)$. $M_\varphi C_f$ is a Hilbert-Schmidt operator on $L^2(G)$ if and only if $f, \varphi \in L^2(G)$.*

Proof. Let $g \in L^2(G)$. Then

$$M_\varphi C_f g(t) = \varphi(t) \int f(t-s) g(s) ds.$$

Thus, $M_\varphi C_f$ is an integral operator whose kernel is the function $k(t,s) = \varphi(t)f(t-s)$. We recall (see Theorem B.9) that an integral operator is Hilbert-Schmidt if and only if its kernel is square integrable. Since we have

$$\begin{aligned}\int\int \mid k(t,s)\mid^2 dtds &= \int(\int \mid f(t-s)\mid^2 dt)\mid \varphi(s)\mid^2 ds \\ &= \int(\int \mid f(t)\mid^2 dt)\mid \varphi(s)\mid^2 ds = \parallel f \parallel_2^2 \parallel \varphi \parallel_2^2,\end{aligned}$$

the assertion of Theorem 3.1 follows. $\square$

For $1 \leq p < \infty$, we denote by $\overline{L^p(G) \cap L^\infty(G)}^\infty$ the closure of $L^p(G) \cap L^\infty(G)$ in $L^\infty(G)$ endowed with its usual topology.

Lemma 3.2. *For $1 \leq p \leq \infty$, we have*

$$\overline{L^p(G) \cap L^\infty(G)}^\infty = \overline{L^1(G) \cap L^\infty(G)}^\infty.$$

Proof. Since one inclusion is obvious, it suffices to show that for each $p \geq 1$ we have

$$L^p(G) \cap L^\infty(G) \subset \overline{L^1(G) \cap L^\infty(G)}^\infty.$$

Set $\mathcal{C} = \overline{L^1(G) \cap L^\infty(G)}^\infty$, and define the involution $(*)$ on $L^\infty(G)$ by: $f^*(t) = \overline{f(t)}$. With this setting, $\mathcal{C}$ is a subC*-algebra of $L^\infty(G)$, see Definition B.11. Let $\psi \in L^p(G) \cap L^\infty(G)$, then $\mid \psi \mid^p \in L^1(G) \cap L^\infty(G) \subset \mathcal{C}$. Using Proposition B.13, we conclude that $\psi \in \mathcal{C}$, and this finishes the proof. $\square$

The proof of the following lemma, about the operators C_f and M_φ, is straightforward.

Lemma 3.3. *Let $f, g \in L^1(G)$ and let $\varphi, \psi \in L^\infty(G)$. Then* *(i)* $\| M_\varphi \| = \| \varphi \|_\infty$ *and* $\| C_f \| = \| \widehat{f} \|_\infty \leq \| f \|_1$.

(ii) $M_\varphi M_\psi = M_{\varphi\psi}$ *and* $C_f C_g = C_{f*g}$.

(iii) $(M_\varphi)^* = M_{\overline{\varphi}}$ *and* $(C_f)^* = C_{f^*}$ *where* $f^*(t) = \overline{f(-t)}$ *a.e.*

The result of the following theorem is not new, but I use new techniques to prove it.

Theorem 3.4. *If* $f \in L^1(G)$ *and* $\varphi \in \overline{L^p(G) \cap L^\infty(G)}^\infty$, *then the product-convolution operator* $M_\varphi C_f$ *is compact on* $L^2(G)$.

Proof. Let $\varphi \in L^2(G) \cap L^\infty(G)$ and $f \in L^1(G)$. Let $(f_n)_{n>0}$ be an approximate identity of $L^1(G)$, consisting of compactly supported functions. Since $\varphi, f_n \in L^2(G) \cap L^1(G)$ for each integer $n > 0$, we conclude, by Theorem 3.1, that the operator $M_\varphi C_{f_n}$ is Hilbert-Schmidt, and thus, is compact. Since $\mathcal{LC}(L^2(G))$ is a sided ideal of $\mathcal{L}(L^2(G))$ (see Proposition B.2), we conclude that $(M_\varphi C_f C_{f_n})_{n>0}$ is a sequence of compact operators. By using Lemma 3.3, we have

$$\| M_\varphi C_f C_{f_n} - M_\varphi C_f \| = \| M_\varphi (C_f C_{f_n} - C_f) \| = \| M_\varphi (C_{f*f_n} - C_f) \|$$
$$\leq \| M_\varphi \| \| C_{f*f_n} - C_f \| \leq \| f * f_n - f \|_1 \| \varphi \|_\infty .$$

If we remember that $(f_n)_{n>0}$ is an approximate identity for $L^1(G)$ and let n tend to infinity, we conclude that the sequence $(M_\varphi C_f C_{f_n})_{n>0}$, of compact operators, converges to the operator $M_\varphi C_f$. Since $\mathcal{LC}(L^2(G))$ is a Banach space, we conclude that the operator $M_\varphi C_f$ is compact. Thus, we have shown that if $\varphi \in L^2(G) \cap L^\infty(G)$, then the operator $M_\varphi C_f$ is compact for each $f \in L^1(G)$. Now let $\varphi \in \overline{L^2(G) \cap L^\infty(G)}^\infty$. There exists a sequence $(\varphi_n)_{n>0}$ such that $\varphi_n \in L^2(G) \cap L^\infty(G)$

for each integer $n > 1$ and $\lim \parallel \varphi_n - \varphi \parallel_\infty = 0$. Then, for each $f \in L^1(G)$, the sequence $(M_{\varphi_n} C_f)_{n>0}$ is contained in $\mathcal{LC}(L^2(G))$. By using Lemma 3.3, we obtain

$$\parallel M_{\varphi_n} C_f - M_\varphi C_f \parallel \leq \parallel M_{\varphi_n} - M_\varphi \parallel \parallel C_f \parallel = \parallel \varphi_n - \varphi \parallel_\infty \parallel C_f \parallel .$$

Thus, the sequence $M_{\varphi_n} C_f$ of compact operators converges to $M_\varphi C_f$ and hence the operator $M_\varphi C_f$ is compact. Finally, the proof can be completed by using Lemma 3.2. □

Corollary 3.5. *If $f \in L^1(G)$ and φ is a measurable function vanishing at infinity, then the product-convolution operator $M_\varphi C_f$ is compact on $L^2(G)$.*

Proof. Since φ is a measurable function vanishing at infinity, for each integer $n > 0$, there exists a compact set $K_n \subset G$ such that

$$| \varphi(t) | < \frac{1}{n} \ \text{ for each } \ t \in (G \setminus K_n).$$

Consider the sequence $(\varphi_n = \varphi 1_{k_n})_{n>0}$, where 1_{k_n} is the characteristic function of K_n. Then $\parallel \varphi_n - \varphi \parallel < \frac{1}{n}$ for each integer $n > 0$, and thus $\varphi_n \to \varphi$ in $L^\infty(G)$. For each integer n the function $\varphi_n \in L^2(G) \cap L^\infty(G)$, since φ_n is compactly supported. Consequently, the function φ is an element of $\overline{L^2(G) \cap L^\infty(G)}^\infty$. This fact along with Theorem 3.4 give Corollary 3.5. □

In [6] R.C.Busby and A.H.Smith obtained the following result.

Theorem 3.6. *Let $\varphi \in L^\infty(\mathbb{R})$. The product-convolution operator $M_\varphi C_f$ on $L^2(\mathbb{R})$ is compact for each $f \in L^1(\mathbb{R})$ if and only if*

$$\lim_{n\to\infty} \int_n^{n+1} | \varphi(t) | \, dt = 0.$$

We remark that Theorem 3.4 is a consequence of the last theorem for the case $G = \mathbb{R}$. The version of Theorem 3.6 for locally compact groups is given in [6], but requires too many preliminaries to be stated, here.

3.2 Applications

Corollary 3.7. *Let $f \in L^1([0,1])$. The operator*

$$T : g \to T(g), \quad \textit{where} \;\; T(g)(t) = \int_0^t f(t-s)g(s)ds \qquad a.e,$$

is compact on $L^2([0,1])$.

Proof. Consider the extension operator $E : L^2([0,1]) \to L^2(\mathbb{R})$ and the restriction operator $R : L^2(\mathbb{R}) \to L^2([0,1])$ which are defined as follows. If $x \in L^2([0,1])$ then $E(x) = X$ is such that $X(t) = x(t)$ for almost all $t \in [0,1]$ and X vanishes out of $[0,1]$. If $X \in L^2(\mathbb{R})$ then $R(X) = x$ is such that $x(t) = X(t)$ for almost all $t \in [0,1]$.

Let $F = E(f)$ and $\varphi = \mathbf{1}_{[0,1]}$, the characteristic function of $[0,1]$. By Corollary 3.5, the operator $M_\varphi C_F M_\varphi$ is compact. It is easy to see that the operators E and R are bounded and that $T = RM_\varphi C_f M_\varphi E$. Therefore, the operator T is compact. $\square$

Remark 3.8. *A result similar to Corollary 3.7 can be stated for any bounded interval $[a,b]$.*

The operators defined in Corollary 3.7 are called Volterra convolution type integral operators [22]. One historically important example of them is the following.

Example 3.9. *The Abel transform of index α, where α is a positive real number,* *is defined by*

$$A_\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds, \quad \textit{for each} \ \ x \in L^2([0,1]),$$

where $\Gamma(\alpha)$ *is the Gamma function [27]. Sometimes* $A_\alpha x$ *is called a Riemann-Liouville integral and it is related with fractional derivatives ([20] and [22]). We claim that the operator* A_α *is compact for any positive real number* α*. Indeed, the function* $f_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ *is integrable on* $[0,1]$*; and thus, the compactness of the operator* A_α *follows by using Corollary 3.7. We note that the operator* A_α *is not a Hilbert-Schmidt operator if* $0 < \alpha \leq \frac{1}{2}$*.*

Proposition 3.10. *Let* ω *be a positive weight function such that the function* $\frac{1}{\omega}$ *is bounded and satisfies the condition of Theorem 3.6. If* $h \in L^1(G)$ *is such that* $h(t) = \overline{h(-t)}$ *(a.e), then there exist a sequence* $(\lambda_n)_{n>0}$ *of nonzero real numbers, an orthonormal system* $(f_n)_{n>0}$ *in* $L^2_\omega(G)$*, and an orthonormal system* $(g_n)_{n>0}$ *in* $L^2_{\omega^{-1}}(G)$*, such that*

$$h * x = \sum_{n>0} \lambda_n < x, f_n >_\omega g_n \quad \textit{for every} \ \ x \in L^2_\omega(G) \quad (1),$$

where the right hand side of (1) is understood to be convergent in the topology of bounded operators, i.e., $\lim_N \sup_{\|x\|_{2,\omega}=1} \| \sum_{n=0}^{N} \lambda_n < x, f_n >_\omega g_n - Tx \|_{2,\omega^{-1}} = 0.$

As an example, think of the Beurling weight $\omega(t) = 1 + \mid t \mid$ defined in the first chapter.

Proof. Let $\varphi = \frac{1}{\sqrt{\omega}}$. Then φ satisfies the condition of Theorem 3.6, and the operator $M_\varphi C_h M_\varphi$ is compact. Since $h = h^*$ and φ is real valued , by Lemma 3.3, we have

$$(M_\varphi C_h M_\varphi)^* = (M_\varphi)^* (C_h)^* (M_\varphi)^* = M_{\overline{\varphi}} C_{h^*} M_{\overline{\varphi}} = M_\varphi C_h M_\varphi.$$

Therefore, the operator $M_\varphi C_h M_\varphi$ is compact and self-adjoint. By the spectral decomposition Theorem for self-adjoint compact operators, see Theorem B.6, there exist a sequence $(\lambda_n)_{n>0}$ of nonzero eigenvalues of $M_\varphi C_f M_\varphi$ and an associated othonormal system $(e_n)_{n>0}$ of eigenvectors such that

$$\varphi(h * \varphi y) = \sum_{n>0} \lambda_n < y, e_n > e_n \text{ for every } \ y \in L^2(G).$$

For each $n > 0$ set $x = \varphi y$, $f_n = \varphi e_n$ and $g_n = \frac{e_n}{\varphi}$. Then we obtain

$$h * \varphi x = \sum_{n>0} \lambda_n < x, f_n >_\omega g_n \text{ for every } \ x \in L^2(G).$$

Since $< f_m, f_n >_\omega = \int f_m \overline{f_n} \omega dt = \int e_m \overline{e_n} dt$, $(f_n)_{n>0}$ is an orthonormal system in $L^2_\omega(G)$. Since $< g_m, g_n >_{\omega-1} = \int g_m \overline{g_n} \omega^{-1} dt = \int e_m \overline{e_n} dt$, $(g_n)_{n>0}$ is an orthonormal system in $L^2_{\omega-1}(G)$. This finishes the proof of Proposition 3.10. □

Corollary 3.11. *Let ω be a positive weight function such that the function $\frac{1}{\omega}$ is bounded and satisfies the condition of Theorem 3.6. If $h \in L^1(G)$, then there exist a sequence $(\lambda_n)_{n>0}$ of complex numbers, a sequence $(f_n)_{n>0}$ in $L^2_\omega(G)$, a sequence $(g_n)_{n>0}$ in $L^2_{\omega^{-1}}(G)$, such that $(f_n)_{n>0}$ is a frame for its span, $(g_n)_{n>0}$ is a frame for its span, and*

$$h * x = \sum_{n>0} \lambda_n < x, f_n >_\omega g_n \ \textit{for every} \ \ x \in L^2_\omega(G), \quad (2)$$

where the right hand side of (2) is understood to be convergent in the topology of bounded operators.

Proof. Consider the two functions $h_1(t) = \frac{1}{2}[h(t) + \overline{h(-t)}]$ and $h_2(t) = \frac{i}{2}[h(t) - \overline{h(-t)}]$. Then $h_1 = h_1^*$, $h_2 = h_2^*$, and $h = h_1 - ih_2$. By Proposition 3.10, there exist two sequences $(\lambda_n^1)_{n>0}$ and $(\lambda_n^2)_{n>0}$, two orthonormal systems $(f_n^1)_{n>0}$ and $(f_n^2)_{n>0}$ in $L^2_\omega(G)$, and two orthonormal systems $(g_n^1)_{n>0}$ and $(g_n^2)_{n>0}$ in $L^2_{\omega^{-1}}(G)$ such that

$$h * x = h_1 * x - ih_2 * x = \sum_{n>0} \lambda_n^1 < x, f_n^1 >_\omega g_n^1 - i\lambda_n^2 < x, f_n^2 >_\omega g_n^2.$$

Define the sequences $(\lambda_n)_{n>0}$, $(f_n)_{n>0}$, and $(g_n)_{n>0}$ as follows:

$$\lambda_{2n+1} = \lambda_n^1 \quad \text{and} \quad \lambda_{2n} = -i\lambda_n^2.$$

$$f_{2n+1} = f_n^1 \quad \text{and} \quad f_{2n} = f_n^2.$$

$$g_{2n+1} = \lambda_n^1 \quad \text{and} \quad g_{2n} = g_n^2.$$

Since each sequence of $(f_n)_{n>0}$ and $(g_n)_{n>0}$ is a frame for its span, as each sequence is the union of two orthonormal systems [17], the proof of Corollary 3.11 is complete.

□

In communication systems a transmitted signal x passes though a channel H and arrives at the receiver as $y = Hx$. If the channel is time invariant then it can be modeled by a convolution operator: $y = Hx = h * x$, [35].

The problem is to recover the transmitted signal x by using the data of the received signal y, in other words we want to solve the equation

$$y = h * x \quad (E),$$

where y is known and x is unknown. Assume that $Hx = 0$ only if $x = 0$, i.e., the operator H is injective. One way to solve (E) is to use Fourier transforms to obtain $\widehat{h}\widehat{x} = \widehat{y}$, i.e., $\widehat{x} = \frac{\widehat{y}}{\widehat{h}}$ (the injectivity of H implies that $\widehat{h}(\gamma) \neq$ for almost all γ). It can be shown that x lies in a Beurling weighted space $L^2_\omega(G)$, see [33], where

$\omega(t) = e^{|t|^\delta}$. If we suppose that $h = h^*$ then, by using Proposition 3.10, equation (E) can be replaced by

$$\sum_{n>0} \lambda_n < x, f_n >_\omega g_n = y \quad (E'),$$

which is a discrete form of equation (E). Since (g_n) is an orthonormal basis for $L^2_{\omega^{-1}}(G)$, we have

$$\lambda_n < x, f_n >_\omega = < y, g_n >_{\omega^{-1}} \quad \text{for every} \quad n > 0,$$

and hence,

$$< x, f_n >_\omega = \frac{< y, g_n >_{\omega^{-1}}}{\lambda_n} \quad \text{for every} \quad n > 0.$$

Remember that $(\lambda_n)_{n>0}$ is a sequence of eigenvalues, hence, $\lambda_n \neq 0$ for all $n > 0$. Therefore, we recover x by the formula

$$x = \sum_{n>0} < x, f_n >_\omega f_n = \sum_{n>0} \frac{< y, g_n >_{\omega^{-1}}}{\lambda_n} f_n.$$

For each integer N consider the finite linear system

$$\sum_{n>0}^{N} \lambda_n < z, f_n >_\omega g_n = y. \quad (S_N)$$

As before S_N has a unique solution x_N that can be written

$$x_N = \sum_{n>0}^{N} < x_N, f_n >_\omega g_n = \sum_{n>0}^{N} \frac{< y, g_n >_{\omega^{-1}}}{\lambda_n} g_n.$$

The sequence $(x_N)_{N>0}$ converges to x in $L^2_\omega(G)$, hence, x_N can be taken as an approximation of the solution x.

3.3 Spectral synthesis of product-convolution operators

A nonzero complex number λ is said to be a characteristic value of a linear operator T, if $\frac{1}{\lambda}$ is an eigenvalue for T. The following proposition is a new result permitting the spectral synthesis of some product-convolution operators.

Proposition 3.12. *Let $h \in L^1(\mathbb{R})$ and $\varphi, \psi \in L^\infty(\mathbb{R})$. Let $x \in L^2(\mathbb{R})$ and assume that the function φ and $h * \psi x$ are twice differentiable on an open set $\Omega \subset \mathbb{R}$. Consider on $L^2(\mathbb{R})$ the operator $H = M_\varphi C_h M_\psi$. Then:*

(i) If x is an eigenfunction of the operator $H = M_\varphi C_h M_\psi$ associated with a characteristic value λ, then x is, on Ω, a solution of the integro-differential equation:

$$E_\lambda : \ \varphi^2 y'' - 2\varphi'\varphi y' + (2(\varphi')^2 - \varphi''\varphi)y = \lambda\varphi^3(h * \psi y)''.$$

(ii) If x is a solution of E_λ, then $\lambda H(x) - x$ is a solution, on Ω, of the differential equation

$$E'_\lambda : \ \varphi^2 y'' - 2\varphi'\varphi y' + (2(\varphi')^2 - \varphi''\varphi)y = 0.$$

Proof. On the open set Ω we have

$$(H(x))' = \varphi(h * \psi x)' + \varphi'(h * \psi x);$$

$$(H(x))'' = \varphi(h * \psi x)'' + 2\varphi'(h * \psi x)' + \varphi''(h * \psi x).$$

Then

$$\varphi^2(H(x))'' - 2\varphi'\varphi(H(x))' = \varphi^3(h * \psi x)'' + (\varphi''\varphi - 2(\varphi')^2)H(x),$$

i.e.,

$$\varphi^2(H(x))'' - 2\varphi'\varphi(H(x))' + (2(\varphi')^2 - \varphi''\varphi)H(x) = \varphi^3(h * \psi x)''.$$

If $x = \lambda H(x)$, the last equality becomes

$$\varphi^2 x'' - 2\varphi'\varphi x' + (2(\varphi')^2 - \varphi''\varphi)x = \lambda\varphi^3(h * \psi x)''.$$

Hence, we obtain (i).

Now suppose that x is a solution of E_λ. Then

$$\varphi^2(\lambda H(x) - x)'' - 2\varphi'\varphi(\lambda H(x) - x)' + (2(\varphi')^2 - \varphi''\varphi)(\lambda H(x) - x) = 0.$$

Hence, we obtain (ii). $\square$

Proposition 3.13. *Let $h = e^{-|t|}$ and let $\varphi, \psi \in L^\infty(\mathbb{R})$. Assume that φ is twice differentiable on some open set $\Omega \subset \mathbb{R}$, and ψ is continuous on Ω. Consider on $L^2(\mathbb{R})$ the operator $H = M_\varphi C_h M_\psi$. If x is an eigenfunction of the operator H associated with a characteristic value λ, then*

(i) The function x is twice differentiable on the open set Ω.

(ii) The function x is, on Ω, a solution of the differential equation

$$\varphi^2 x'' - 2\varphi'\varphi x' + (2(\varphi')^2 - \varphi''\varphi - \varphi^2 + 2\lambda\varphi^3\psi)x = 0.$$

Proof. We have $x = \lambda H x = \lambda\varphi(h*\psi x)$. Since $h, \psi x \in L^2(\mathbb{R})$, the function $(h*\psi x)$ is continuous, see appendix A.3. Therefore, the function $x = \lambda\lambda\varphi(h*\psi x)$ is continuous on Ω, since the functions φ and $h * \psi x$ are continuous on Ω. Now let $t \in \Omega$. Then we have

$$(h * \psi x)(t) = \int e^{-|t-s|}\psi(s)x(s)ds = e^{-t}\int_{-\infty}^{t} e^{s}\psi(s)x(s)ds + e^{t}\int_{t}^{\infty} e^{-s}\psi(s)x(s)ds.$$

Since Ω is open, there exists an open interval (a, b) such that $t \in (a, b) \subset \Omega$. We have

$$\int_{-\infty}^{t} e^{s}\psi(s)x(s)ds = \int_{-\infty}^{a} e^{s}\psi(s)x(s)d + \int_{a}^{t} e^{s}\psi(s)x(s)ds.$$

The function $e^s\psi(s)x(s)$ is continuous on (a,b). Then by the Fundamental Theorem of Calculus the function $t \to \int_a^t e^s\psi(s)x(s)ds$ is differentiable on (a,b) and its derivative at t is given by $e^t\psi(t)x(t)$. By using a similar arguments we can show that the function $t \to \int_t^\infty e^{-s}\psi(s)x(s)ds$ is differentiable on Ω and its derivative at t is given by $e^{-t}\psi(t)x(t)$. Therefore, the function $(h*\psi x)$ is differentiable on Ω. By computing the derivatives, we obtain

$$(h*\psi x)'(t) = -e^{-t}\int_{-\infty}^t e^s\psi(s)x(s)ds + e^t\int_t^\infty e^{-s}\psi(s)x(s)ds.$$

By proceeding as before, we can show that $(h*\psi x)$ is twice differentiable on Ω; and, by computing the derivatives, we obtain

$$(h*\psi x)''(t) = h*\psi x - 2\psi x. \tag{3.1}$$

Now by using Proposition 3.12, we have

$$\varphi^2 x'' - 2\varphi'\varphi x' + (2(\varphi')^2 - \varphi''\varphi)x = \lambda\varphi^3(h*\psi x)''.$$

And by using (3.1), we obtain

$$\varphi^2 x'' - 2\varphi'\varphi x' + (2(\varphi')^2 - \varphi''\varphi - \varphi^2 + 2\lambda\varphi^3\psi)x = 0.$$

This finishes the proof. $\square$

Example 3.14. *Let $h(t) = e^{-|t|}$ and $\varphi = \mathbf{1}_{[0,1]}$. We have $h(t) = \overline{h(-t)}$ and φ is real valued. Then the operator $H = M_\varphi C_f M_\varphi$ is self-adjoint and is Hilbert-Schmidt, since $h, \varphi \in L^2(\mathbb{R})$. Then the spectrum of H is countable and consists of nonzero real numbers.*

We claim that the operator $x \to h * x$ is injective on $L^2(\mathbb{R})$. Indeed, Let $x \in L^2(\mathbb{R})$ such that $h * x = 0$. Then $\widehat{h * x} = \widehat{h}\widehat{x} = 0$. However

$$\widehat{h}(\gamma) = \int e^{-2\pi i t} e^{-|t|} dt = \frac{2}{1 + 4\pi^2\gamma^2}$$

is never vanishing. Then $\widehat{x} = 0$ and thus $x = 0$. This shows the claim.

Let $H_0 = RHE$ where $E : L^2([0,1]) \to L^2(\mathbb{R})$ is the extension operator and $R : L^2(\mathbb{R}) \to L^2([0,1])$ is the restriction operator. Then the operator H_0 is compact self-adjoint and injective on $L^2([0,1])$. By Proposition 3.13, the eigenfunctions of H_0 satisfy the differential equation

$$E_\lambda : y'' + (2\lambda - 1)y = 0.$$

If $\lambda = \frac{1}{2}$, the solutions of E_λ are given by $y(t) = at + b$, and

$$\frac{1}{2}H_0(y)(t) = y(t) + \frac{1}{2}[(a-b)e^{-t} - \frac{2a+b}{e}e^t].$$

Therefore, $\frac{1}{2}$ is not a characteristic value for H_0.

If $\lambda \neq \frac{1}{2}$, the solutions of E_λ are given by $y(t) = ae^{\mu t} + be^{-\mu t}$, where $\mu^2 = 1 - 2\lambda$. We have

$$\lambda H_0(y)(t) = y(t) + \frac{1}{2}[(a(\mu-1) - b(\mu+1))e^{-t} + \frac{-a(\mu+1)e^\mu + b(\mu-1)e^{-\mu}}{e}e^t].$$

Thus, λ is a characteristic value of H_0 if and only if

$$b = a\frac{\mu-1}{\mu+1}, \tag{3.2}$$

$$e^\mu = \mp\frac{\mu-1}{\mu+1}. \tag{3.3}$$

The solutions of the equation $e^\mu = \frac{\mu-1}{\mu+1}$, are given by the purely imaginary numbers $\mu = i\alpha$ such that $\tan(\frac{\alpha}{2}) = \frac{1}{\alpha}$ and $\alpha > 0$.

The solutions of the equation $e^\mu = -\frac{\mu-1}{\mu+1}$*, are given by the purely imaginary numbers* $\mu = i\alpha$ *such that* $\tan(\frac{\alpha}{2}) = -\alpha$ *and* $\alpha > 0$*.*

Now we shall give the spectral decomposition of the operator H_0*. First recall that* H_0 *is compact, self-adjoint and injective. By Theorem B.6, there exists an orthonormal basis* $(e_n)_{n\geq 0}$ *of* $L^2([0,1])$ *and a sequence* $(\eta_n)_{n\geq 0}$ *such that*

$$Tx = \sum \eta_n < x, e_n > e_n = \sum < Tx, e_n > e_n.$$

Define the sequence $(\alpha_n)_{n\geq 0}$ *as follows:*

$$\tan(\frac{\alpha_{2(n+1)}}{2}) = \frac{1}{\alpha_{2(n+1)}} \quad \text{and } \alpha_{2(n+1)} \in ((2n+3)\pi, (2n+5)\pi),$$
$$\tan(\frac{\alpha_{2n+1}}{2}) = -\alpha_{2n+1} \quad \text{and } \alpha_{2n+1} \in ((2n+1)\pi, (2n+3)\pi),$$
$$\tan(\frac{\alpha_0}{2}) = \frac{1}{\alpha_0} \quad \text{and } \alpha_0 \in (0, \pi).$$

The sequence of eigenvalues of H_0 *is* $(\frac{2}{1+\alpha_n^2})_{n\geq 0}$*, and the spectral decomposition of* H_0 *is given by*

$$H_0(x) = \sum_{n>0} \frac{2}{1+\alpha_n^2} < x, e_n > e_n,$$

where $(e_n)_{n\geq 0}$*, defined by*

$$e_n(t) = \sqrt{\frac{1+\alpha_n^2}{2(3+\alpha_n^2)}}[e^{i\alpha_n t} + (-1)^n e^{i\alpha_n} e^{-i\alpha_n t}],$$

is the orthonormal basis of $L^2([0,1])$ *consisting of the eigenfunctions of* H_0*.*

Example 3.15. *Let* $h(t) = e^{-|t|}$ *and* $\varphi(t) = e^{\alpha t}\mathbf{1}_{(-\infty,0)}$*,* $\alpha > 0$*. Let* $H_0 = RHE$ *where* $E : L^2((-\infty,0)) \to L^2(\mathbb{R})$ *is the extension operator and* $R : L^2(\mathbb{R}) \to L^2((-\infty,0))$ *is the restriction operator. As in Example 3.14 we can show that* *the operator H_0 is Hilbert-Schmidt, self-adjoint and injective on $L^2((-\infty,0))$. The eigenfunctions of H_0 satisfy the differential equation*

$$E_\lambda : y'' - 2\alpha y' - (\alpha^2 - 1 + 2\lambda e^{2\alpha t})y = 0.$$

The solution of E_λ are given by:

$$e^{\alpha t} J_{\frac{1}{\alpha}}(\frac{\mu}{2} e^{\alpha t})$$

where $J_{\frac{1}{\alpha}}$ is the Bessel function of order $\frac{1}{\alpha}$, see [27], and $\mu^2 = 2\lambda$. By calculations similar to Example 3.14, we can deduce that the sequence of eigenvalues is given by the sequence $(\frac{2}{\mu_n^2})_{n\geq 0}$, where $\frac{\mu_n}{2}$ are the zeros of the Bessel function $J_{\frac{1}{\alpha}-1}$, and the sequence of eigenfunctions is given by

$$e_n(t) = \frac{\sqrt{2\alpha}}{J_{\frac{1}{\alpha}}(\frac{\mu_n}{2})} e^{\alpha t} J_{\frac{1}{\alpha}}(\frac{\mu_n}{2} e^{\alpha t}).$$

The spectral decomposition of H_0 is given by

$$H_0(f) = \sum_{n>0} \frac{4\alpha}{(\mu_n J_{\frac{1}{\alpha}}(\frac{\mu_n}{2}))^2} < f, e_n > e_n.$$

Example 3.16. *Let $h(t) = e^{-|t|}$ and $\varphi(t) = \frac{1}{\sqrt{t}} \mathbf{1}_{(1,\infty)}$. We note that φ is zero at infinity, hence, the operator $H = M_\varphi C_f M_\varphi$ is compact by Corollary 3.5, but not necessarily Hilbert-Schmidt. Let $H_0 = RHE$ where $E : L^2((1,\infty)) \to L^2(\mathbb{R})$ is the extension operator and $R : L^2(\mathbb{R}) \to L^2((1,\infty))$ is the restriction operator. As before, we can show that H_0 is compact, self-adjoint, and injective. The eigenfunctions of H satisfy the differential equation*

$$E_\lambda : t^2 y'' + ty' - (t^2 - 2\lambda t + \frac{1}{4})y = 0.$$

To solve E_λ we set $y(t) = \frac{1}{\sqrt{t}} e^t z(-2t)$ to obtain the equation

$$F_\lambda : tz'' - tz' - \lambda z = 0,$$

which is a degenerate hypergeometric equation, [27].

3.4 Summary

Product-convolution operators are studied in this chapter.

In Theorem 3.4, I proved that if φ belongs to the closure of $L^p(G) \cap L^\infty(G)$ in $L^\infty(G)$ and if $f \in L^1(G)$, then the product-convolution operator $M_\varphi C_f$ is compact on $L^2(G)$. This is a known result, for which I gave a new proof. In section 3.2, I presented some applications of the results of section 3.1. Proposition 3.10 showed that a convolution operator $C_h : L^2_\omega(G) \to L^2_{\omega^{-1}}(G)$ is compact, if ω is such that ω^{-1} satisfies the condition of Theorem 3.4. I used this new result and the spectral decomposition theorem for compact operators to solve equations of the forms $y = h * x$. These kinds of equations recall equations arising in communications theory [35]. In Proposition 3.12 and Proposition 3.13, I proved that the eigenfunctions of some product-convolution operators can be obtained as solutions of some differential equations. This new result is interesting, since it gives the spectral decomposition of some compact operators which are not necessarily Hilbert-Schmidt operators. As an illustration, I ended this chapter by three examples. Incidentally, I obtain some special functions as eigenfunctions of some product-convolution operators. I also obtained the zeros of some special functions as eigenvalues of some product-convolution operators.

Chapter 4

The tensor product of frames

It is known that the tensor product of two orthonormal bases is an orthonormal basis. In this chapter, I prove the following new result.

Theorem 4.26. *The sequence $(f_i^k)_{i\in\mathcal{I}_k}$ is a frame (Riesz basis) for a Hilbert space $\mathcal{H}_k$, $k \in \{1, 2\}$, if and only if $(f_i^1 \otimes f_j^2)_{(i,j)\in\mathcal{I}_1\times\mathcal{I}_2}$ is a frame (Riesz basis) for $\mathcal{H}_1 \otimes \mathcal{H}_2$.*

This result improves a result by C.Heil, J.Ramanathan, and P.Topiwala [19]. They prove that the tensor product of a frame with itself is a frame. Section 4.1 and appendix B contain the essentials of operator theory needed for Chapter 4. In Section 4.1, I describe $\mathcal{H}_1 \otimes \mathcal{H}_2$, the tensor product of two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. In section 4.2, I prove Theorem 4.8 and Theorem 4.12, two new contributions to the theory of tensor products. In Section 4.3, I define frames and state some of their properties. I prove Lemma 4.8, this new result is an interesting connection between the theory of frames and the theory of operators. In section 4.4, I prove Theorem 4.26, the main result of this Chapter. I use this result to extend the Lyubarski and Seip-Wallsten theorem, characterizing Gabor frames generated by the Gaussian function, to higher dimensions([17], [24], and [32]). Section 4.5 is a summary of all results of this chapter.

4.1 The tensor product of Hilbert spaces

The main reference for this section is [14]. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. If $k \in \{1, 2\}$, we denote by $\| \, . \, \|_k$ and $<,>_k$ the norm and the inner product of $\mathcal{H}_k$, respectively.

Definition 4.1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. An operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ is called Hilbert-Schmidt, if for some orthonormal basis $(e_i)_{i \in \mathcal{I}}$ in $\mathcal{H}_1$ one has*

$$\sum_{i \in \mathcal{I}} \| T(e_i) \|_2^2 < \infty.$$

We denote by $\mathcal{L}_2(\mathcal{H}_1, \mathcal{H}_2)$ the space of all Hilbert-Schmidt operators: $\mathcal{H}_1 \to \mathcal{H}_2$.

Proposition 4.2. *Let $T \in \mathcal{L}_2(\mathcal{H}_1, \mathcal{H}_2)$.*

(i) The series $\sum_{i \in \mathcal{I}} \| T(e_i) \|_2^2$ is independent of the orthonormal basis $(e_i)_{i \in \mathcal{I}}$ used. Thus, we can define

$$\| T \|_{H(\mathcal{H}_1, \mathcal{H}_2)} = \Big(\sum_{i \in \mathcal{I}} \| T(e_i) \|_2^2\Big)^{\frac{1}{2}}.$$

(ii) If $T^* : \mathcal{H}_2 \to \mathcal{H}_1$ is the adjoint operator of T, then T^* is Hilbert-Schmidt and

$$\| T^* \|_{H(\mathcal{H}_2, \mathcal{H}_1)} = \| T \|_{H(\mathcal{H}_1, \mathcal{H}_2)} .$$

(iii) The operator T is compact and we have $\| T \|_{O(\mathcal{H}_1, \mathcal{H}_2)} \leq \| T \|_{H(\mathcal{H}_1, \mathcal{H}_2)}$, where $\| T \|_{O(\mathcal{H}_1, \mathcal{H}_2)}$ is the operator norm of T.

(iv) If X and Y are two Hilbert spaces, $S : X \to \mathcal{H}_1$ a bounded operator, and $R : \mathcal{H}_2 \to Y$ a bounded operator. Then the operator $RTS : X \to Y$ is Hilbert-Schmidt. Furthermore, we have

$$\| RTS \|_{H(X,Y)} \leq \| R \|_{O(\mathcal{H}_2, Y)} \| T \|_{H(\mathcal{H}_1, \mathcal{H}_2)} \| S \|_{O(X, \mathcal{H}_1)} .$$

Example 4.3. *For $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_2$ we denote by $E_{g,f} : \mathcal{H}_1 \to \mathcal{H}_2$ the rank one operator, defined by $E_{g,f}(x) =< x, f >_1 g$ for each $x \in \mathcal{H}_1$. The operator $E_{g,f}$ is Hilbert-Schmidt, and $\| E_{g,f} \|_{H(\mathcal{H}_1,\mathcal{H}_2)}=\| f \|_1 \| g \|_2$. Any finite combination of rank one operators, is called a finite rank operator and is also Hilbert-Schmidt.*

Proposition 4.4. *(i) For every $F_1, F_2 \in \mathcal{L}_2(\mathcal{H}_1, \mathcal{H}_2)$ the series $\sum_{i\in\mathcal{I}} < F_1(e_i), F_2(e_i) >_2$ is absolutely convergent and independent of the particular orthonormal basis used to define it; we hence define*

$$< F_1, F_2 >= \sum_{i\in\mathcal{I}} < F_1(e_i), F_2(e_i) >_2 .$$

(ii) The map $(F_1, F_2) \to < F_1, F_2 >$ defines an inner product on $\mathcal{L}_2(\mathcal{H}_1, \mathcal{H}_2)$, and with this inner product $\mathcal{L}_2(\mathcal{H}_1, \mathcal{H}_2)$ is a Hilbert space.

(iii) The map $T \to T^*$ is an isometric bijective antilinear map: $\mathcal{L}_2(\mathcal{H}_1, \mathcal{H}_2) \to L_2(\mathcal{H}_2, \mathcal{H}_1)$.

Theorem 4.5. *The topological tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be interpreted as the Hilbert space $\mathcal{L}_2(\mathcal{H}_2, \mathcal{H}_1)$.*

The interpretation of Theorem 4.1 is based on the identification $f \otimes g \simeq E_{f,g}$. I shall use this theorem as a definition of $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Remark 4.6. *By Proposition 4.4(iii) we have the identification $\mathcal{L}_2(\mathcal{H}_1, \mathcal{H}_2) \simeq \mathcal{L}_2(\mathcal{H}_2, \mathcal{H}_1)$, hence $\mathcal{H}_1 \otimes \mathcal{H}_2 \simeq \mathcal{H}_2 \otimes \mathcal{H}_1$.*

4.2 The tensor product of bounded operators

If X and Y are two Banach spaces, we denote by $\mathcal{L}(X,Y)$ the space of all bounded operators: $X \to Y$. The norm of each $T \in \mathcal{L}(X,Y)$ shall be denoted by $\parallel T \parallel_{O(X,Y)}$. If $X = Y$ we denote $\mathcal{L}(X) = \mathcal{L}(X,Y)$ and $\parallel T \parallel_{O(X)} = \parallel T \parallel_{O(X,Y)}$. If $F_1 \in \mathcal{L}(\mathcal{H}_1)$, $F_2 \in \mathcal{L}(\mathcal{H}_2)$ and $H \in L_2(\mathcal{H}_2, \mathcal{H}_1)$, then, by Proposition 4.2(iv), $F_1 H F_2^* \in L_2(\mathcal{H}_2, \mathcal{H}_1)$. We define the operator $F_1 \otimes F_2 : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ by the rule $F_1 \otimes F_2(H) = F_1 H F_2^*$. For example, for every $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_2$, we have

$$F_1 \otimes F_2(f \otimes g) = F_1 E_{f,g} F_2^* = E_{F_1(f), F_2(g)} = F_1(f) \otimes F_2(g).$$

Proposition 4.7. *(i) If $F_1 \in \mathcal{L}(\mathcal{H}_1)$ and $F_2 \in \mathcal{L}(\mathcal{H}_2)$, then the operator $F_1 \otimes F_2 \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ and $\parallel F_1 \otimes F_2 \parallel_{O(\mathcal{H}_1 \otimes \mathcal{H}_2)} = \parallel F_1 \parallel_{O(\mathcal{H}_1)} \parallel F_2 \parallel_{O(\mathcal{H}_2)}$.*

(ii) If $F_1, G_1 \in \mathcal{L}(\mathcal{H}_1)$ and $F_2, G_2 \in \mathcal{L}(\mathcal{H}_2)$,then

$$(F_1 \otimes F_2)(G_1 \otimes G_2) = F_1 G_1 \otimes F_2 G_2.$$

For a proof of Proposition 4.7 we refer to [14].

Let X be a Banach space and $(F_N)_{N>0}$ be a sequence in $\mathcal{L}(X)$. We say that $(F_N)_{N>0}$ converges in the strong operator topology to F, if $F_N(x) \to F(x)$ for each $x \in X$. Under the strong operator topology, $\mathcal{L}(X)$ is complete. The following theorem is an new result.

Theorem 4.8. *Let $(F_N)_{N>0}$ be a bounded sequence in $\mathcal{L}(\mathcal{H}_1)$ and $(G_N)_{N>0}$ be a bounded sequence in $\mathcal{L}(\mathcal{H}_2)$. If the sequence $(F_N)_{N>0}$ converges in the strong operator topology to $F \in \mathcal{L}(\mathcal{H}_1)$ and the sequence $(G_N)_{N>0}$ converges in the strong operator* *topology to $G \in \mathcal{L}(\mathcal{H}_2)$, then the sequence $(F_N \otimes G_N)_{N>0}$ converges in the strong operator topology to $F \otimes G$.*

From now on, I shall denote by $\| H \|$ and $< H, K >$ the norm and inner product, respectively, for $\mathcal{L}_2(\mathcal{H}_2, \mathcal{H}_1)$.

Proof. There exist two constants C_1 and C_2 such that:

$$\lim_{N\to\infty} \| F_N(f) - F(f) \|_1 = 0 \quad \text{and} \quad \| F_N(f) \| \leq C_1 \quad \forall f \in \mathcal{H}_1;$$

$$\lim_{N\to\infty} \| G_N(g) - G(g) \|_2 = 0 \quad \text{and} \quad \| G_N(g) \| \leq C_2 \quad \forall g \in \mathcal{H}_2.$$

For each $H \in \mathcal{H}_2 \otimes \mathcal{H}_1 = \mathcal{L}_2(\mathcal{H}_2, \mathcal{H}_1)$, we have

$$\begin{aligned} \| F_N \otimes G_N(H) - F \otimes G(H) \| &= \| F_N H G_N^* - F H G^* \| \\ &= \| (F_N H - F H) G_N^* + F(H G_N^* - H G^*) \| \\ &\leq \| F_N H - F H \| \| G_N^* \| + \| F \| \| H G_N^* - H G^* \| \\ &\leq C_2 \| F_N H - F H \| + \| F \| \| H G_N^* - H G^* \| . \end{aligned}$$

If $(g_n)_{n>0}$ is an orthonormal basis of $\mathcal{H}_1$, then

$$\| F_N H - F H \|^2 = \sum_{n>0} \| F_N H(g_n) - F H(g_n) \|_1^2 . \tag{4.1}$$

Since the sequence $(F_N)_{N>0}$ converges in the strong operator topology to F, then

$$\lim_{N\to\infty} \| F_N H(g_n) - F H(g_n) \|_1^2 = 0 \quad \text{for each } n > 0. \tag{4.2}$$

By Proposition 4.2(iv), we have

$$\| F_N H - F H \| \leq \| F_N - F \|_{O(\mathcal{H}_1)} \| H \| \leq 2C_1 \| H \| .$$

In the other hand, we have

$$\| H \|^2 = \sum_{n>0} \| H(g_n) \|^2 .$$

Then, by (4.1), we obtain

$$\sum_{n>0} \| F_N H(g_n) - FH(g_n) \|_1^2 \leq 4C_1^2 \sum_{n>0} \| H(g_n) \|^2 < \infty. \tag{4.3}$$

Since (4.2) and (4.3) are satisfied, by Lebesgue's Dominated Theorem we obtain

$$\lim_{N\to\infty} \| F_N H - FH \| = 0.$$

Similarly, we can show that

$$\lim_{N\to\infty} \| HG_N^* - HG^* \| = 0.$$

Therefore,

$$\lim_{N\to\infty} \| F_N \otimes G_N(H) - F \otimes G(H) \| = 0.$$

We conclude, that the sequence $(F_N \otimes G_N)_{N>0}$ converges in the strong operator topology to $F \otimes G$. $\square$

The following lemma is a new result.

Lemma 4.9. *For each $k \in \{1, 2\}$ let F_k be a nonzero bounded operator on $\mathcal{H}_k$, f_k be a unit vector such that $F_k(f_k) \neq 0$, $U_k : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_k$ and $V_k : \mathcal{H}_k \to \mathcal{H}_1 \otimes \mathcal{H}_2$, defined for each $f \in \mathcal{H}_1$, $g \in \mathcal{H}_2$, and $H \in \mathcal{H}_1 \otimes \mathcal{H}_2$, by $U_1(H) = H(F_2(f_2))$, $V_1(f) = E_{f,f_2}$, $U_2(H) = H^*(F_1(f_1))$, and $V_2(g) = E_{f_1,g}$. Then:*

(i) $\| U_1 \|_{O(\mathcal{H}_1\otimes\mathcal{H}_2,\mathcal{H}_1)} \leq \| F_2(f_2) \|_2$, $\| U_2 \|_{O(\mathcal{H}_1\otimes\mathcal{H}_2,\mathcal{H}_2)} \leq \| F_1(f_1) \|_1$, and the operators V_1 and V_1 are isometric.

(ii) $U_1[F_1 \otimes F_2]V_1 =\| F_2(f_2) \|_2^2 F_1$ *and* $U_2[F_1 \otimes F_2]V_2 =\| F_1(f_1) \|_1^2 F_2$.

(iii) $U_kV_k =< F_k(f_k), f_k > I_{\mathcal{H}_k}$, *for each* $k \in \{1, 2\}$.

Proof. If $H \in \mathcal{H}_1 \otimes \mathcal{H}_2$, then $\| U_1(H) \|_1=\| H(F_2(f_2)) \|_1\leq\| H \|\| F_2(f_2) \|_2$; and $\| U_2(H) \|_2=\| H^*(F_1(f_1)) \|_1\leq\| H^* \|\| F_1(f_1) \|_2=\|_1\leq\| H \|\| F_1(f_1) \|_2$. Thus we obtain (i)

(ii) If $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_2$, then

$$[U_1(F_1 \otimes F_2)V_1](f) = [U_1(F_1 \otimes F_2)](E_{f,f_2}) = U_1(E_{F_1(f),F(f_2)})$$

$$= E_{F_1(f),F_2(f_2)}(F_2(f_2)) =\| F_2(f_2) \|_2^2 F_1(f).$$

And

$$[U_2(F_1 \otimes F_2)V_2](f) = [U_2(F_1 \otimes F_2)](E_{f_1,g}) = U_2(E_{F_1(f_1),F_2(g)})$$

$$=\| F_1(f_1) \|_1^2 F_2(f).$$

Thus, we obtain (ii).

(iii) If $f \in \mathcal{H}_1$, then

$$U_1V_1(f) = U_1(E_{f,f_2}) = E_{f,f_2}(F_2(f_2)) =< F_2(f_2), f_2 >_2 f.$$

If $g \in \mathcal{H}_2$, then

$$U_2V_2(g) = U_2(E_{f_1,g}) = E_{g,f_1}(F_1(f_1)) =< F_1(f_1), f_1 >_1 g.$$

This completes the proof. □

Remark 4.10. *The operators* U_1 *and* V_1 *are linear while the operators* U_2 *and* V_2 *are antilinear.*

In the following Lemma I summarize some facts that I shall use later. The proof of this lemma is straightforward.

Lemma 4.11. *(i) If $(f,g) \in \mathcal{H}_1 \times \mathcal{H}_2$, then $(E_{f,g})^* = E_{g,f}$.*

(ii) Let $f, f' \in \mathcal{H}_1 \setminus \{0\}$ and $g, g' \in \mathcal{H}_2 \setminus \{0\}$. If $E_{f,g} = E_{f',g'}$, then there exist two nonzero constants a and b such that $f' = af$ and $g' = bg$.

(iii) If $(f,g) \in \mathcal{H}_1 \times \mathcal{H}_2$ and $(F,G) \in \mathcal{L}(\mathcal{H}_1) \times \mathcal{L}(\mathcal{H}_2)$, then

$$FE_{f,g} = E_{F(f),g} \quad \text{and} \quad E_{f,g}G = E_{f,G^*(g)}.$$

(iv) If $(f,g) \in \mathcal{H}_1 \times \mathcal{H}_2$ and $H \in \mathcal{H}_1 \otimes \mathcal{H}_2 = \mathcal{L}_2(\mathcal{H}_1, \mathcal{H}_2)$, then

$$< H, E_{f,g} >=< H(g), f >_1=< g, H^*(f) >_2 .$$

(v) If $H \in \mathcal{H}_1 \otimes \mathcal{H}_2$ and $f_k, f'_k \in \mathcal{H}_k$ for each $k \in \{1,2\}$, then

$$E_{f_1,f'_1} H E_{f_2,f'_2} =< H, E_{f'_1,f'_2} > E_{f_1,f_1}.$$

The following theorem is a new contribution to the theory of the tensor product.

Theorem 4.12. *The operator $F_1 \otimes F_2$ is invertible in $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ if and only if the operator F_k is invertible in $\mathcal{L}(\mathcal{H}_k)$ for each $k \in \{1,2\}$.*

Proof. Suppose that the operator F_k is invertible in $\mathcal{L}(\mathcal{H}_k)$ for each $k \in \{1,2\}$. By Proposition 4.7(ii), we have

$$(F_1 \otimes F_2)(F_1^{-1} \otimes F_2^{-1}) = (F_1F_1^{-1} \otimes F_2F_2^{-1}) = I_{\mathcal{H}_1} \otimes I_{\mathcal{H}_2} = I_{\mathcal{H}_1 \otimes \mathcal{H}_2}.$$

Similarly we have $(F_1^{-1} \otimes F_2^{-1})(F_1 \otimes F_2) = I_{\mathcal{H}_1 \otimes \mathcal{H}_2}$. Therefore, the operator $F_1 \otimes F_2$ is invertible in $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ and

$$(F_1 \otimes F_2)^{-1} = F_1^{-1} \otimes F_2^{-1}.$$

Conversely, suppose the operator $F_1 \otimes F_2$ is invertible in $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Then $F_k \neq 0$ for each $k \in \{1, 2\}$. Let f_k be a unit vector such that $F_k(f_k) \neq 0$. Let U_1, V_1, U_2, and V_2 be the operators defined in Lemma 4.11. Consider the operators:

$$U_1' : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \text{ given by } U_1'(H) = H(f_2),\ H \in \mathcal{H}_1 \otimes \mathcal{H}_2;$$

$$V_1' : \mathcal{H}_1 \to \mathcal{H}_1 \otimes \mathcal{H}_2 \text{ given by } V_1'(f) = E_{f, F_2(f_2)},\ f \in \mathcal{H}_1;$$

$$U_2' : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_2 \text{ given by } U_2'(H) = H^*(f_1),\ H \in \mathcal{H}_1 \otimes \mathcal{H}_2;$$

$$V_2' : \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2 \text{ given by } V_2'(f) = E_{F_1(f_1), g},\ g \in \mathcal{H}_2.$$

Let

$$\begin{aligned} F_1' &= \frac{1}{\| f_2 \|_2^2} U_1' [F_1 \otimes F_2]^{-1} V_1'; \\ F_2' &= \frac{1}{\| f_1 \|_1^2} U_2' [F_1 \otimes F_2]^{-1} V_2'. \end{aligned}$$

First, we observe that the operators F_1' and F_2' are bounded. Now let $f \in \mathcal{H}_1$, hence

$$\begin{aligned} F_1' F_1(f) &= \frac{1}{\| f_2 \|_2^2} U_1' [F_1 \otimes F_2]^{-1} V_1'(F_1(f)) \\ &= \frac{1}{\| f_2 \|_2^2} U_1' [F_1 \otimes F_2]^{-1} (E_{F_1(f), F_2(f_2)}). \end{aligned}$$

By Lemma 4.11(i), we have $F_1 \otimes F_2(E_{f, f_2}) = E_{F_1(f), F_2(f_2)}$. Therefore,

$$F_1' F_1(f) = \frac{1}{\| f_2 \|_2^2} U_1'(E_{f, F_2 f_2}) = \frac{1}{\| f_2 \|_2^2} E_{f, f_2}(f_2) = f.$$

Similarly, we can show that $F_2'F_2(g) = g$ for each $g \in \mathcal{H}_2$. Thus,

$$F_1'F_1 = I_{\mathcal{H}_1} \ \text{ and } \ F_2'F_2 = I_{\mathcal{H}_2}. \tag{4.4}$$

By using Proposition 4.7(ii), we obtain

$$(F_1' \otimes F_2')(F_1 \otimes F_2) = (F_1'F_1 \otimes F_2'F_2) = I_{\mathcal{H}_1} \otimes I_{\mathcal{H}_2} = I_{\mathcal{H}_1 \otimes \mathcal{H}_2}.$$

Hence, $(F_1' \otimes F_2') = (F_1 \otimes F_2)^{-1}$. Consequently, we have

$$(F_1F_1' \otimes F_2F_2') = (F_1 \otimes F_2)(F_1' \otimes F_2') = I_{\mathcal{H}_1 \otimes \mathcal{H}_2}.$$

Therefore, if $(f, g) \in \mathcal{H}_1 \times \mathcal{H}_2 \setminus \{(0, 0)\}$, by using Lemma 4.11(iii), we obtain

$$E_{F_1F_1'(f), F_2F_2'(g)} = (F_1F_1' \otimes F_2F_2')(E_{f,g}) = E_{f,g}.$$

By using Lemma 4.11(ii), there exist two positive constants a and b such that

$$F_1(F_1'(f)) = af \ \text{ and } \ F_2(F_2'(g)) = bg.$$

These equations show that the operators F_1 and F_2 are surjective. By (4.4), the operators F_1 and F_2 are injective. This complete the proof of Theorem 4.8. □

4.3 Frames

All results of this section are stated and proved in [17].

Definition 4.13. *A sequence $(f_i)_{i \in \mathcal{I}}$ is a frame for a Hilbert space $\mathcal{H}$ if there exist constants $A, B > 0$, called frame bounds, such that for all $f \in \mathcal{H}$*

$$A \parallel f \parallel^2 \leq \sum_{i \in \mathcal{I}} |< f, f_i >|^2| \leq B \parallel f \parallel^2 .$$

The largest possible value for A and the smallest possible value for B are called optimal frame bounds. If $A = B$, then we say the frame is tight.

Example 4.14. *An orthonormal basis is a tight frame with frame bounds $A = B = 1$, the union of any two orthonormal bases is a tight frame with frame bounds $A = B = 2$, and the union of an orthonormal basis with n arbitrary unit vectors is a frame with frame bounds $A = 1$ and $B = 1 + n$.*

Frames generalize orthonormal bases. However, these trivial examples already show that in general the frame elements are neither orthogonal to each other nor linearly independent.

Definition 4.15. *For any sequence $(f_i)_{i\in\mathcal{I}}$, the coefficient operator or analysis operator C is given by $C(f) = (< f, f_i >)_{i\in\mathcal{I}}$, the synthesis operator or reconstruction operator D is defined for a finite sequence $(c_j)_{j\in\mathcal{J}}$ by $\sum_{j\in\mathcal{J}} c_j f_j$, and the frame operator S is defined on $\mathcal{H}$ by $S(f) = \sum_{i\in\mathcal{I}} < f, f_i > f_i$.*

Proposition 4.16. *Suppose that $(f_i)_{i\in\mathcal{I}}$ is a frame for $\mathcal{H}$.*

(i) C is a bounded operator from $\mathcal{H}$ into $l^2(\mathcal{I})$ with closed range.

(ii) The operators C and D are adjoint to each other; that is , $D = C^*$. Consequently, D extends to a bounded operator from $l^2(\mathcal{I})$ into $\mathcal{H}$ and satisfies

$$\| \sum_{i\in\mathcal{I}} c_i f_i \|^2 \leq B \sum_{i\in\mathcal{I}} | c_i |^2 .$$

(iii) The frame operator $S = C^*C = DD^*$ maps $\mathcal{H}$ into $\mathcal{H}$ and is a positive invertible operator satisfying $AI_{\mathcal{H}} \leq S \leq BI_{\mathcal{H}}$ and $B^{-1}I_{\mathcal{H}} \leq S^{-1} \leq A^{-1}I_{\mathcal{H}}$. In particular $(f_i)_{i\in\mathcal{I}}$ is a tight frame if and only if $S = AI_{\mathcal{H}}$.

(iv) The optimal frame bounds are $B_{opt} = \parallel S \parallel_{O(\mathcal{H})}$ *and* $A_{opt} = \parallel S^{-1} \parallel_{O(\mathcal{H})}^{-1}$

Definition 4.17. *Let* $\{f_i, i \in \mathcal{I}\}$ *be a countable set in a Banach space* X. *The series* $\sum_{i\in\mathcal{I}} f_i$ *is said to converge unconditionally to* $f \in X$ *if for every* $\varepsilon > 0$ *there exists a finite set* $J \subset \mathcal{I}$ *such that*

$$\parallel f - \sum_{i\in J} f_i \parallel < \varepsilon \quad \text{for all finite sets } K \supseteq J.$$

Proposition 4.18. *Let* $\{f_i, i \in \mathcal{I}\}$ *be a countable set in a Banach space* X. *Then the following are equivalent:*

(i) $f = \sum_{i\in\mathcal{I}} f_i$ *converges unconditionally to* $f \in X$.

(ii) For every enumeration, i.e., a bijective map $\pi : \mathbb{N} \to \mathcal{I}$, *the sequence of partial sums* $\sum_{n=1}^{N} f_{\pi(n)}$ *converges to* $f \in X$.

In particular, the limit f *is independent of the enumeration* π.

Proposition 4.19. *Let* $(f_i)_{i\in\mathcal{I}}$ *be a frame for* $\mathcal{H}$. *If*

$$f = \sum_{i\in\mathcal{I}} c_i f_i \quad \text{and} \quad (c_i)_{i\in\mathcal{I}} \in l^2(\mathcal{I}),$$

then the series $\sum_{i\in\mathcal{I}} c_i f_i$ *converges unconditionally to* $f \in \mathcal{H}$.

Proposition 4.20. *Suppose that* $(f_i)_{i\in\mathcal{I}}$ *is a frame for* $\mathcal{H}$. *Then the following are equivalent:*

(i)The analysis operator C *maps onto* $l^2(\mathcal{I})$.

(ii) There exist constant $A', B' > 0$ *such that the inequalities*

$$A' \sum_{i\in J} \mid c_i \mid^2 \leq \parallel \sum_{i\in J} c_i f_i \parallel^2 \leq B' \sum_{i\in J} \mid c_i \mid^2$$

hold for all finite sequences $(c_i)_{i\in J}$.

(iii) $(f_i)_{i\in\mathcal{I}}$ is the image of an orthonormal basis under an invertible bounded operator.

Any frame satisfying one of the conditions of Proposition 4.20 is called a Riesz basis.

Now let $\mathcal{H} = L^2(\mathbb{R}^d)$, where d is a positive integer. A *translation* of $g \in L^2(\mathbb{R}^d)$ by $a \in \mathbb{R}$ is $T_a g(x) = g(x-a)$, a *modulation* of g by $b \in \mathbb{R}$ is $M_b g(x) = e^{2\pi i b.x} g(x)$, where $b.x$ is the dot product of b and x. Translations and modulations define bijective isometries on $\in L^2(\mathbb{R}^d)$. A composition $T_a M_b$ is called a *time-frequency shift* of g.

Definition 4.21. *Let $g \in L^2(\mathbb{R}^d) \setminus \{0\}$ and $\alpha, \beta > 0$. The Gabor system generated by g, α and β is*

$$\mathcal{G}(g,\alpha,\beta) = \{\ T_{\alpha m} M_{\beta n} g : m, n \in \mathbb{Z}^d\}.$$

If a Gabor system is a frame for $L^2(\mathbb{R}^d)$, then it is called a Gabor frame.

Theorem 4.22 (Lyubarskii and Seip-Wallstén Theorem). *Let $\varphi(x) = 2^{\frac{1}{4}} e^{-x^2}$ be the Gaussian function on $\mathbb{R}$.*

$$\mathcal{G}(\varphi,\alpha,\beta) \ \text{ is a Gabor frame for } \ L^2(\mathbb{R}) \ \text{ if and only if } \ \alpha\beta < 1.$$

4.4 The tensor product of frames

To each pair of sequences $((f_n)_{n>0}, (g_n)_{n>0})$ in a Hilbert space $\mathcal{H}$, we associate a linear operator F defined by

$$F(f) = \sum_{n>0} < f, f_n > g_n, \quad \text{for each} \quad f \in \mathcal{H} \quad \text{such that} \quad F(f) \in \mathcal{H}.$$

The following lemma is new result.

Lemma 4.23. *Let F be the operator associated with the pair $((f_n)_{n>0}, (e_n)_{n>0})$ where $(f_n)_{n>0}$ is a sequence and $(e_n)_{n>0}$ is an orthonormal basis in $\mathcal{H}$.*

(i) The sequence $(f_n)_{n>0}$ is a frame with frame bounds A and B if and only if the operator F is bounded and, for each $f \in \mathcal{H}$, we have

$$A \parallel f \parallel^2 \leq \parallel F(f) \parallel^2 \leq B \parallel f \parallel^2 . \tag{4.5}$$

(ii) Suppose that $(f_n)_{n>0}$ is a frame. The sequence $(f_n)_{n>0}$ is a Riesz basis if and only if the operator F is bijective.

Proof. Suppose that $(f_n)_{n>0}$ is a frame with frame bounds A and B. Then, for each $f \in \mathcal{H}$, we have

$$A \parallel f \parallel^2 \leq \sum_{n>0} |< f, f_n >|^2 \leq B \parallel f \parallel^2 . \tag{4.6}$$

In particular, the series $\sum_{n>0} |< f, f_n >|^2 < \infty$. Therefore, the operator F is defined on $\mathcal{H}$ and for each $f \in \mathcal{H}$ we have

$$\parallel F(f) \parallel^2 = \sum_{n>0} |< f, f_n >|^2 .$$

Thus, (4.5) follows from (4.6).

Conversely, assume that F is bounded and that (4.5) holds for each $f \in \mathcal{H}$. Since for each $f \in \mathcal{H}$: $\parallel F(f) \parallel^2 = \sum_{n>0} |< f, f_n >|^2$, then (4.6) follows from (4.5). This finishes the proof of (i).

(ii) Since $(f_n)_{n>0}$ is a frame then by (i) the operator F is bounded and satisfies (4.5) for each $f \in \mathcal{H}$. The adjoint of F is defined by

$$F^*(f) = \sum_{n>0} < f, e_n > f_n \quad \text{for each} \quad f \in \mathcal{H}.$$

The operator F^* is bounded, since F is; and we have

$$F^*(e_n) = f_n \quad \text{for each} \quad n > 0.$$

If F is bijective then so is F^*. Therefore $(f_n)_{n>0}$ is a Riesz basis, as it is the image of the orthonormal basis $(e_n)_{n>0}$ by the bounded bijective operator F^*.

Conversely, if $(f_n)_{n>0}$ is a Riesz basis, then there exist an orthonormal basis $(g_n)_{n>0}$ and a bounded linear bijection G such that $G(g_n) = f_n$ for each $n > 0$. Since $(e_n)_{n>0}$ and $(g_n)_{n>0}$ are two orthonormal bases, there exists a unitary operator U such that $U(g_n) = e_n$ for each $n > 0$. Then $F^*U(g_n) = f_n$ for each $n > 0$. Therefore, $F^*U = G$ as they coincide on the orthonormal basis $(g_n)_{n>0}$. Therefore, F^* is bijective and hence so is F. This completes the proof. □

Remark 4.24. *If $(f_n)_{n>0}$ is a frame and $(e_n)_{n>0}$ is an orthonormal basis for $\mathcal{H}$, by using Proposition 4.19, the series $\sum_{n>0} < f, f_n > e_n$ is convergent unconditionally to $F(f)$ for each $f \in \mathcal{H}$.*

By using the frame inequalities, we can deduce that the operator F is injective and has a closed range.

Now let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. If $k \in \{1, 2\}$, we denote by $\| \, . \, \|_k$ and $<, >_k$ the norm and the inner product of $\mathcal{H}_k$, respectively. Now I state the main new result of this chapter.

Theorem 4.25. *For each $k \in \{1, 2\}$ let $(f_n^k)_{n>0}$ be a sequence in $\mathcal{H}_k$.*

(i) The sequence $(f_i^1 \otimes f_j^2)_{i,j>0}$ is a frame for $\mathcal{H}_1 \otimes \mathcal{H}_2$ if and only if $(f_n^k)_{n>0}$ is a frame for $\mathcal{H}_k$ for each $k \in \{1, 2\}$.

Moreover, if A_k and B_k are the frame bounds of $(f_n^k)_{n>0}$, $k \in \{1,2\}$, then A_1A_2 and B_1B_2 are the frame bounds of $(f_i^1 \otimes f_j^2)_{i,j>0}$.

(ii) The sequence $(f_i^1 \otimes f_j^2)_{i,j>0}$ is a Riesz basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$ if and only if $(f_n^k)_{n>0}$ is a Riesz basis for $\mathcal{H}_k$ for each $k \in \{1,2\}$.

Proof. (i) Suppose that $(f_i^1 \otimes f_j^2)_{i,j>0}$ is a frame for $\mathcal{H}_1 \otimes \mathcal{H}_2$ with frame bounds A and B. Then, for each $H \in \mathcal{H}_1 \otimes \mathcal{H}_2$, we have

$$A \parallel H \parallel^2 \leq \sum_{i,j>0} |< H, f_i^1 \otimes f_j^2 >|^2 \leq B \parallel H \parallel^2 . \tag{4.7}$$

If $(f,g) \in H \in \mathcal{H}_1 \times \mathcal{H}_2 \setminus \{(0,0)\}$, then

$$\parallel f \otimes g \parallel = \parallel f \parallel_1 \parallel g \parallel_2 . \tag{4.8}$$

We have

$$\begin{aligned} \sum_{i,j>0} |< f \otimes g, f_i^1 \otimes f_j^2 >|^2 &= \sum_{i,j>0} |< f, f_i^1 >_1|^2 |< g, f_j^2 >_2|^2 \\ &= (\sum_{i>0} |< f, f_i^1 >_1|^2)(\sum_{j>0} |< g, f_j^2 >_2|^2). \end{aligned} \tag{4.9}$$

Since $(f,g) \neq (0,0)$, hence, by (4.7), the left most member of (4.9) is finite and nonzero. Therefore each term of the product of the right most member of (4.9) is finite and nonzero. Fix $g \in \mathcal{H}_2 \setminus \{0\}$ and let $f \in \mathcal{H}_1 \setminus \{0\}$. Then, by using (4.7), (4.8) and (4.9), we obtain

$$\frac{A \parallel g \parallel_2^2}{\sum_{j>0} |< g, f_j^2 >_2|^2} \parallel f \parallel_1^2 \leq \sum_{i,j>0} |< f, f_i^1 >_1|^2 \leq \frac{B \parallel g \parallel_2^2}{\sum_{j>0} |< g, f_j^2 >_2|^2} \parallel f \parallel_1^2 .$$

Since the last inequalities are obviously satisfied for $f = 0$, we conclude that $(f_n^1)_{n>0}$ is a frame for $\mathcal{H}_1$. Similarly, we can show that $(f_n^2)_{n>0}$ is a frame for $\mathcal{H}_2$.

Conversely, suppose that $(f_n^k)_{n>0}$ is a frame for $\mathcal{H}_k$ with frame bounds A_k and B_k, $k \in \{1,2\}$. Then, for each $(f,g) \in \mathcal{H}_1 \times \mathcal{H}_2$, we have

$$A_1 \parallel f \parallel_1^2 \leq \sum_{i>0} |< f, f_i^1 >_1|^2 \leq B_1 \parallel f \parallel_1^2 . \tag{4.10}$$

$$A_2 \parallel g \parallel_2^2 \leq \sum_{j>0} |< g, f_j^2 >_2|^2 \leq B_2 \parallel g \parallel_2^2 . \tag{4.11}$$

For each $k \in \{1,2\}$, let $(e_n^k)_{n>0}$ be an orthonormal basis for $\mathcal{H}_k$; and consider

$$F^k(f) = \sum_{n>0} < f, f_n^k > e_n^k, \quad \text{for each} \quad f \in \mathcal{H}_k, \tag{4.12}$$

the bounded operator associated with $((f_n^k)_{n>0}, (e_n^k)_{n>0})$, as defined in the beginning of this section. For each $k \in \{1,2\}$, consider the sequence $(F_N^k)_{N>0}$, of bounded linear operators, defined by

$$F_N^k(f) = \sum_{n=0}^{N} < f, f_n^k > e_n^k, \quad \text{for each} \quad f \in \mathcal{H}_k. \tag{4.13}$$

The sequence $(F_N^k)_{N>0}$ converges in the strong operator topology to F^k and we have $\parallel F_N^k \parallel_{O(\mathcal{H}_k)} \leq \parallel F^k \parallel_{O(\mathcal{H}_k)}$. Hence, by Theorem 4.8, the sequence $(F_N^1 \otimes F_N^2)_{N>0}$ converges in the strong operator topology to $F^1 \otimes F^2$.

On the other hand, we have $F_N^k = \sum_{n=0}^N E_{e_n^k, f_n^k}$ for each $k \in \{1,2\}$. Therefore, for each $H \in \mathcal{H}_1 \otimes \mathcal{H}_2$, we have

$$\begin{aligned} F_N^1 \otimes F_N^2(H) &= F_N^1 H (F_N^2)^* = (\sum_{i=0}^{N} E_{e_i^1, f_i^1}) H (\sum_{j=0}^{N} E_{f_j^2, e_j^2}) \\ &= \sum_{i,j=0}^{N} E_{e_i^1, f_i^1} H E_{f_j^2, e_j^2} = \sum_{i,j=0}^{N} < H, E_{f_i^1, f_j^2} > E_{e_i^1, e_j^2}. \end{aligned}$$

The second equality follows by Lemma 4.11(i) and the last equality is a consequence of Lemma 4.11(v). Thus, we have

$$F_N^1 \otimes F_N^2(H) = \sum_{i,j=0}^{N} < H, f_i^1 \otimes f_j^2 > e_i^1 \otimes e_j^2.$$

Since, the sequence $(F_N^1 \otimes F_N^2)_{N>0}$ converges in the strong operator topology to $F^1 \otimes F^2$, we can write

$$F^1 \otimes F^2(H) = \sum_{i,j>0} < H, f_i^1 \otimes f_j^2 > e_i^1 \otimes e_j^2 \quad \text{for each} \quad H \in \mathcal{H}_1 \otimes \mathcal{H}_2. \qquad (4.14)$$

Hence, $F^1 \otimes F^2$ is the operator associated with the pair $((f_i^1 \otimes f_j^2)_{i,j>0}, (e_i^1 \otimes e_j^2)_{i,j>0})$. By Proposition 4.7, the operator $F^1 \otimes F^2$ is bounded. And since $(e_i^1 \otimes e_j^2)_{i,j>0}$ is an orthonormal basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$, see [14], we obtain

$$\| F^1 \otimes F^2(H) \| = \sum_{i,j>0} |< H, f_i^1 \otimes f_j^2 >|^2, \quad \text{for each} \quad H \in \mathcal{H}_1 \otimes \mathcal{H}_2. \qquad (4.15)$$

Now, for each $H \in \mathcal{H}_1 \otimes \mathcal{H}_2 = \mathcal{L}_2(\mathcal{H}_2, \mathcal{H}_1)$, we have

$$\| F^1 \otimes F^2(H) \|^2 = \| F^1 H (F^2)^* \|^2 = \sum_{j>0} \| F^1 H (F^2)^*(e_j^2) \|_1^2 .$$

Then, by (4.10), we obtain

$$A_1 \sum_{j>0} \| H(F^2)^*(e_j^2) \|_1^2 \leq \| F^1 \otimes F^2(H) \|^2 \leq B_1 \sum_{j>0} \| H(F^2)^*(e_j^2) \|_1^2,$$

i.e.,

$$A_1 \| H(F^2)^* \|_1^2 \leq \| F^1 \otimes F^2(H) \|^2 \leq B_1 \| H(F^2)^* \|_1^2 .$$

By Proposition 4.2(ii) and the identification $\mathcal{H}_1 \otimes \mathcal{H}_2 \simeq \mathcal{H}_2 \otimes \mathcal{H}_1$, we obtain

$$A_1 \| F^2 H^* \|_1^2 \leq \| F^1 \otimes F^2(H) \|^2 \leq B_1 \| F^2 H^* \|_1^2,$$

i.e.,

$$A_1 \sum_{i>0} \parallel F^2 H^*(e_i^1) \parallel_2^2 \leq \parallel F^1 \otimes F^2(H) \parallel^2 \leq B_1 \sum_{i>0} \parallel F^2 H^*(e_i^1) \parallel_2^2 .$$

Therefore, by (4.11), we obtain

$$A_1 A_2 \sum_{i>0} \parallel H^*(e_i^1) \parallel_2^2 \leq \parallel F^1 \otimes F^2(H) \parallel^2 \leq B_1 B_2 \sum_{i>0} \parallel H^*(e_i^1) \parallel_2^2,$$

i.e.,

$$A_1 A_2 \parallel H^* \parallel^2 \leq \parallel F^1 \otimes F^2(H) \parallel^2 \leq B_1 B_2 \parallel H^* \parallel^2 .$$

Thus, by Proposition 4.2(ii) and the identification $\mathcal{H}_1 \otimes \mathcal{H}_2 \simeq \mathcal{H}_2 \otimes \mathcal{H}_1$, we have

$$A_1 A_2 \parallel H \parallel^2 \leq \parallel F^1 \otimes F^2(H) \parallel^2 \leq B_1 B_2 \parallel H \parallel^2 .$$

Finally, by using (4.15), we obtain

$$A_1 A_2 \parallel H \parallel^2 \leq \sum_{i,j>0} |< H, f_i^1 \otimes f_j^2 >|^2 \leq B_1 B_2 \sum_{i>0} \parallel H \parallel^2 .$$

This shows that $(f_i^1 \otimes f_j^2)_{i,j>0}$ is a frame for $\mathcal{H}_1 \otimes \mathcal{H}_2$ with frame bounds $A_1 A_2$ and $B_1 B_2$.

(ii) For each $k \in \{1, 2\}$, let $(f_n^k)_{n>0}$ be a frame and $(e_n^k)_{n>0}$ an orthonormal basis for $\mathcal{H}_k$. By (i), $(f_i^1 \otimes f_j^2)_{i,j>0}$ is a frame for $\mathcal{H}_1 \otimes \mathcal{H}_2$. Let F^k be the operator associated with $((f_n^k)_{n>0}, (e_n^k)_{n>0})$, for each $k \in \{1, 2\}$. Then $F^1 \otimes F^2$ is the operator associated with $((f_i^1 \otimes f_j^2)_{i,j>0}, (e_i^1 \otimes e_j^2)_{i,j>0})$, as was shown in (i). By Theorem 4.7, the operator $F^1 \otimes F^2$ is bijective if and only if F^k is for each $k \in \{1, 2\}$. Therefore, statement (ii) follows by using Lemma 4.23. □

Since all series, involved in the proof of Theorem 4.25, either had positive terms or converged unconditionally, we can restate Theorem 4.25 in a more general form as follows.

Theorem 4.26. *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two countable sets. For each $k \in \{1, 2\}$ let $(f_i^k)_{i\in\mathcal{I}_k}$ be a sequence in $\mathcal{H}_k$.*

(i) The sequence $(f_i^1 \otimes f_j^2)_{(i,j)\in\mathcal{I}_1\times\mathcal{I}_2}$ is a frame for $\mathcal{H}_1 \otimes \mathcal{H}_2$ if and only if $(f_i^k)_{i\in\mathcal{I}_k}$ is a frame for $\mathcal{H}_k$ for each $k \in \{1, 2\}$.

Moreover, if A_k and B_k are the frame bounds for $(f_i^k)_{i\in\mathcal{I}_k}$, $k \in \{1, 2\}$, then A_1A_2 and B_1B_2 are the frame bounds for $(f_i^1 \otimes f_j^2)_{(i,j)\in\mathcal{I}_1\times\mathcal{I}_2}$.

(ii) The sequence $(f_i^1 \otimes f_j^2)_{(i,j)\in\mathcal{I}_1\times\mathcal{I}_2}$ is a Riesz basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$ if and only if $(f_i^k)_{i\in\mathcal{I}_k}$ is a Riesz basis for $\mathcal{H}_k$ for each $k \in \{1, 2\}$.

The following result was conjectured by I. Daubechies and A. Grossmann [7] and then was proved independently by Y. Lyubarskii [24] and K. Seip and R. Wallstén [32].

> **Lyubarskii and Seip-Wallstén Theorem.** *Let $\varphi(x) = 2^{\frac{1}{4}}e^{-\pi x^2}$ be the Gaussian function on $\mathbb{R}$.*
>
> $$\mathcal{G}(\varphi, \alpha, \beta) \ \text{ is a frame for } \ L^2(\mathbb{R}) \ \text{ if and only if } \ \alpha\beta < 1.$$

The following corollary is a new result extending Lyubarskii and Seip-Wallstén theorem to higher dimensions.

Corollary 4.27. *Let $\varphi_d(x) = 2^{\frac{d}{4}}e^{-|x|^2}$ be the Gaussian function on $\mathbb{R}^d$.*

$$\mathcal{G}(\varphi_d, \alpha, \beta) \ \text{ is a Gabor frame for } \ L^2(\mathbb{R}^d) \ \text{ if and only if } \ \alpha\beta < 1.$$

Proof. Since $L^2(\mathbb{R}^{d_1}) \otimes L^2(\mathbb{R}^{d_2}) \simeq L^2(\mathbb{R}^{d_1+d_2})$, see [14]. And obviously we have $\varphi_{d_1} \otimes \varphi_{d_2} = \varphi_{d_1+d_2}$. Therefore, Corollary 4.27 is a consequence of Theorem 4.26 and the Lyubarskii and Seip-Wallstén theorem. $\square$

4.5 Summary

In this chapter I proved the following theorem.

Theorem 4.26. *The sequence* $(f_i^k)_{i\in\mathcal{I}_k}$ *is a frame (Riesz basis) for a Hilbert space* $\mathcal{H}_k$, $k \in \{1, 2\}$, *if and only if* $(f_i^1 \otimes f_j^2)_{(i,j)\in\mathcal{I}_1\times\mathcal{I}_2}$ *is a frame (Riesz basis) for* $\mathcal{H}_1 \otimes \mathcal{H}_2$.

This new result improves a result by C.Heil, J.Ramanathan, and P.Topiwala [19]. They prove that the tensor product of a frame with itself is a frame. Incidentally, I proved two new contributions to theory of tensor products. The first contribution is Theorem 4.8 concerning the convergence of the tensor product of two convergent sequences. The second contribution is Theorem 4.12, where I proved that the tensor product of two bounded operators is bijective if and only if each part of this tensor product is a bounded bijective operator. To prove Theorem 4.12 I used Lemma 4.9, a new synthesis lemma. Lemma 4.23 is a new result giving an interesting connection between the theory of frames and the theory of operators. In order to prove Theorem 4.26, I used Theorem 4.8, Theorem 4.12, and Lemma 4.23. Using Theorem 4.26, I was able to extend the Lyubarskii and Seip-Wallstén theorem to higher dimensions. This new result is stated in Corollary 4.27.

Appendix A

Harmonic analysis on locally compact abelian groups

The main reference of this appendix is [31]. On every *locally compact abelian group* G there exists a nonnegative regular measure, unique up to a positive constant, the so called *Haar measure* of G, which is translation invariant. From now on, we choose one Haar measure of G that we shall denote by dx. When we say an integrable function, we mean integrable with respect to the measure dx. If E is a measurable set, we denote by $\mid E \mid$ the measure of E by dx. Two measurable functions f and g are said to be *equal almost everywhere*, if $\mid \{t : f(t) \neq g(t)\} \mid = 0$. We denote $f = g$ a.e. This gives an equivalence relation on the set of measurable functions. We shall always identify a function with its class modulo a.e. A function f is said to be *essentially bounded*, if there exists a constant $M \geq 0$ such that

$$\mid f(x) \mid \leq M \text{ a.e.} \quad (1.1)$$

The lowest M satisfying (1.1) is called *the essential bound* of f, denoted by $\| f \|_\infty = M$. For every $p \in [1, \infty)$, we denote

$$L^p(G) = \{\text{measurable functions } f : \| f \|_p =: (\int \mid f(t) \mid^p dt)^{\frac{1}{p}} < \infty\};$$

$$\text{and } L^\infty(G) = \{\text{measurable functions } f : \| f \|_\infty < \infty\}.$$

For every $p \in [1, \infty)$, we denote by p' the real number satisfying $\frac{1}{p} + \frac{1}{p'} = 1$.

A.1. *For every $p \in [1, \infty]$, the space $L^p(G)$ is a Banach space. For every $p \in [1, \infty)$ the dual of the space $L^p(G)$ is the space $L^{p'}(G)$.*

The *translation* by $x \in G$ of a measurable function f is defined by the formula $\iota_x f(t) = f(t-x)$.

A.2. *(i) For each $x \in G$ the translation operator $f \to \iota_x f$ is a bijective isometry on $L^p(G)$, $1 \leq p \leq \infty$.*

(ii) Let $p \in [1,\infty)$ and $f \in L^p(G)$. The map $x \to \iota_x f$ is uniformly continuous from G into $L^p(G)$.

A.3. The convolution. *The convolution of two measurable functions f and g, if it exists a.e, is defined as follows: $(f * g)(x) = \int f(x-y)g(y)dy$ a.e.*

(i) Let $p \in [1,\infty]$. If $f \in L^1(G)$ and $g \in L^p(G)$, then $f * g \in L^p(G)$, and we have: $\| f * g \|_p \leq \| f \|_1 \| g \|_p$.

(ii) If $f, g \in L^1(G)$, then $f * g \in L^1(G)$. Under its space operations and the convolution, $L^1(G)$ is a commutative Banach algebra. If G is discrete, $L^1(G)$ has a unit.

(iii) Let $p \in [1,\infty)$. If $f \in L^p(G)$ and $g \in L^{p'}(G)$, then $f * g \in \mathcal{C}_0(G)$, the space of continuous functions vanishing at infinity.

A.4. Characters. *A complex function γ is called a character of G if $|\gamma(t)| = 1$ and $\gamma(s+t) = \gamma(s)\gamma(t)$ for all $t, s \in G$.*

A.5. The dual group. *The set of all continuous characters of G forms an abelian group $\widehat{G}$, the so called dual group of G, if addition is defined by*

$$(\gamma_1 + \gamma_2)(t) = \gamma_1(t)\gamma_2(t), \quad t \in G \quad \text{and} \quad \gamma_1, \gamma_2 \in \widehat{G}.$$

A topology and a Haar measure, that we shall denote by $d\gamma$, can be defined on $\widehat{G}$ to make it a locally compact abelian group.

A.6. The Fourier transform. *The Fourier transform of an $f \in L^1(G)$ is a measurable function $\widehat{f}$ on $\widehat{G}$, defined by*

$$\widehat{f}(\gamma) = \int f(t)\overline{\gamma(t)}dt \ \ a.e.$$

It can be shown that $\widehat{f}$ is continuous and vanishes at infinity.

A.7. *If $f, g \in L^1(G)$, then $\widehat{(f * g)} = \widehat{f}\widehat{g}$.*

A.8. *The map $f \to \widehat{f}$ is an isometry from $L^1(G) \cap L^2(G)$ into $L^2(\widehat{G})$. By continuity this map can be extended to an isomeric bijection $L^2(G) \to L^2(\widehat{G})$. We denote by $\widehat{f}$, and we call it the Fourier transform of f, the image of each $f \in L^2(G)$ by this last isometry.*

A.9. Bounded measures. *A measure μ is said to be bounded on G, if*

$$\| \mu \| =: \sup_{f \in C_0(G), \|f\|_\infty \leq 1} \left| \int f d\mu \right| < \infty.$$

The set $M(G)$ of all bounded measures on G is a Banach space. It is the topological dual of $\mathcal{C}_0(G)$. The space $L^1(G)$ can be seen as a subspace of $M(G)$, if we identify each $f \in L^1(G)$ with the bounded measure μ_f defined for each measurable set E by $\mu_f(E) = \int \mathbf{1}_E d\mu_f$, where $\mathbf{1}_E$ is the characteristic function of E.

A.10. *Let $(\mu_n)_{n>0}$ be a sequence of bounded measures. We say that $(\mu_n)_{n>0}$ converges weakly or vaguely to a bounded measure μ, if*

$$\lim_n \int f d(\mu_n - \mu) = 0 \ \ for\ all \ \ f \in C_0(G).$$

Every bounded sequence of bounded measures has a subsequence weakly convergent.

A.11. *If μ is a bounded measure, the total variation of μ is a bounded measure $|\mu|$ defined for all measurable set E by*

$$|\mu|(E) = \sup \sum |\mu(E_i)|,$$

the supremum is taken over all finite collections of pairwise disjoint Borel sets E_i whose union is E. It can be shown that $\|\mu\| = \int d|\mu|$.

Let $\lambda, \mu \in M(G)$ and $\mu \times \lambda$ be their product measure on the product group $G^2 = G \times G$. Associate with each Borel set E in G the set $E_2 = \{(s,t) : s + t \in E\}$. Then E_2 is a Borel set of G^2. We define $\mu * \lambda$ by

$$(\mu * \lambda)(E) = (\mu \times \lambda)(E_2).$$

A.12. *With its space operations and the convolution, $M(G)$ is a commutative Banach algebra with unit. The unit of $M(G)$ is the δ measure.*

A.13. *Let $p \in [1, \infty]$. If $\mu \in M(G)$ and $f \in L^p(G)$, then $\mu * g \in L^p(G)$, and we have $\|\mu * f\|_p \leq \|\mu\| \|f\|_p$.*

Appendix B

Operator theory

In this section we summarize some facts from operator theory. Our main reference is [14].

In the following, we let $\mathcal{H}$ be a Hilbert space and denote its inner product by $<,>$. The norm on $\mathcal{H}$ is defined by $\| f \|= (< f, f >)^{\frac{1}{2}}$. We denote by $\mathcal{L}(\mathcal{H})$ the space of bounded operators on $\mathcal{H}$; and define, on $\mathcal{L}(\mathcal{H})$, the following norm

$$\| A \|= \sup_{x\neq 0} \frac{\| Ax \|}{\| x \|} = \sup_{\|x\|<1} \| Ax \|= \sup_{\|x\|=1} \| Ax \|,$$

called the norm of bounded operators. With this norm, $\mathcal{L}(\mathcal{H})$ is a Banach algebra.

Definition B.1. *A linear operator T on $\mathcal{H}$ is said to be compact, if $T(B)$ is relatively compact in $\mathcal{H}$, where B is the unit ball of $\mathcal{H}$. We denote by $\mathcal{LC}(\mathcal{H})$ the set of all compact operators on $\mathcal{H}$.*

Proposition B.2. *Under the norm of bounded operators, $\mathcal{LC}(\mathcal{H})$ is a closed sided ideal of $\mathcal{L}(\mathcal{H})$. Moreover if $T \in \mathcal{LC}(\mathcal{H})$ then so is T^*, the adjoint operator of T.*

Definition B.3. *Let $T \in \mathcal{L}(\mathcal{H})$.*

(i) A complex number λ is said to be a spectral value of T, if the operator $(T - \lambda I)$ has no inverse in $\mathcal{L}(\mathcal{H})$, where I is the unit operator of $\mathcal{H}$. We denote by $\sigma(T)$ the set of all spectral values of T.

(ii) A complex number λ is said to be an eigenvalue of T, if there exists a

nonzero vector x such that $Tx = \lambda x$. The vector x is then called an eigenvector of T and λ is the eigenvalue associated to it.

Proposition B.4. *If $T \in \mathcal{L}(\mathcal{H})$, then $\sigma(T)$ is a nonempty compact subset of $\mathbb{C}$.*

Theorem B.5. *Let T be a compact operator. Then*

(i) $\sigma(T)$ is a nonempty compact and at most countable subset of $\mathbb{C}$.

(ii) All elements of $\sigma(T)$, with a possible exception of zero, are isolated.

(iii) If λ is a nonzero spectral value of T, then λ is an eigenvalue.

Theorem B.6 (The spectral decomposition theorem for compact and self-adjoint operator)**.** *For any compact and self adjoint-operator T of $\mathcal{H}$, there exist a sequence of eigenvalues $(\lambda_n)_{n\geq 1}$ and a corresponding sequence of eigenvectors $(e_n)_{n\geq 1}$, such that we have*

(i) All terms of the sequence $(\lambda_n)_{n\geq 1}$ are real.

(ii) $|\lambda_1| \geq |\lambda_2| \geq ...... \geq |\lambda_n| \geq ...... > 0$; and if the sequence $(\lambda_n)_{n\geq 1}$ is infinite then $\lambda_n \to 0$.

(iii) The sequence of eigenvectors $(e_n)_{n\geq 1}$ is an orthonormal system.

(iv) The sequence of eigenvectors $(e_n)_{n\geq 1}$ is an orthonormal basis if and only if the operator T is injective.

(v) For every $x \in \mathcal{H}$ we have

$$Tx = \sum \lambda_n < x, e_n > e_n = \sum < Tx, e_n > e_n.$$

Definition B.7. *A linear operator T is called Hilbert Schmidt on $\mathcal{H}$, or HS, if for some orthonormal basis $(e_n)_{n\geq 1}$ for $\mathcal{H}$ one has $\sum \parallel T(e_n) \parallel^2 < \infty$. We denote by $\mathcal{L}_2(\mathcal{H})$ the space of all HS operator on $\mathcal{H}$.*

If $(f_n)_{n\geq 1}$ is another orthonormal basis, it can be shown that

$$\sum \| T(f_n) \|^2 = \sum \| T(e_n) \|^2 < \infty.$$

If T is a HS operator, we define its norm to be $\|T\|_{HS} = (\sum \| T(e_n) \|^2)^{\frac{1}{2}}$, which is independent of the orthonormal basis (e_n) used . $\mathcal{L}_2(\mathcal{H})$ with this norm is a Banach algebra. If $T, S \in \mathcal{L}_2(\mathcal{H})$ and (e_n) is an orthonormal basis, then the formula

$$< f, g >_{HS} = \sum < T(e_n), S(e_n) >$$

is independent of the orthonormal basis (e_n) used, and defines an inner product on $\mathcal{L}_2(\mathcal{H})$. Endowed with this inner product, $\mathcal{L}_2(\mathcal{H})$ is a Hilbert space.

Proposition B.8. *(i) A HS operator is a compact operator on $\mathcal{H}$. Moreover, HS operators are dense in $\mathcal{LC}(\mathcal{H})$ with respect to the topology of bounded operators.*

(ii) $\mathcal{L}_2(\mathcal{H})$ is a sided ideal,i.e., a left and right sides ideal, of $\mathcal{L}(\mathcal{H})$. Moreover, we have

$$\|TA\|_{HS}, \|AT\|_{HS} \leq \|A\|\|T\|_{HS}, \quad \textit{for each} \quad T \in \mathcal{L}_2(\mathcal{H}) \quad \textit{and for each} \quad A \in \mathcal{L}(\mathcal{H}).$$

Now let $\mathcal{H} = L^2(G)$, the Hilbert space of square integrable functions on a locally compact group G. Let $k(x, y)$ be a measurable function on $G \times G$. The formula $f \to \int k(x, y)f(y)dy$ defines a linear mapping K from some subspace $\mathcal{D}$ of $L^2(G)$, where the integral makes sense, called an *integral operator.* The function $k(x, y)$ is called *the kernel* of the integral operator K.

Theorem B.9. *Let k be a measurable function on $G \times G$. The integral operator*

$$f \to \int k(x, y)f(y)dy$$

defines a HS operator on the Hilbert space $L^2(G)$ if and only if the kernel $k \in L^2(G \times G)$.

Definition B.10. *Let A be an algebra over the field $\mathbb{C}$ of complex numbers. An involution on A is a map $x \to x^*$ from A into itself such that:*

(i) $(x^*)^* = x$, *for each* $x \in A$.

(ii) $(x + y)^* = x^* + y^*$, *for each* $x, y \in A$.

(iii) $(\lambda x)^* = \overline{\lambda} x^*$, *for each* $x \in A$.

(iv) $(xy)^* = y^* x^*$, *for each* $x, y \in A$.

Definition B.11. *A C^*-algebra is a Banach algebra A together with an involution $x \to x^*$, such that*

(i) $\| x^* \| = \| x \|$, *for each* $x \in A$.

(ii) $\| x x^* \| = \| x \|^2$, *for each* $x \in A$.

Definition B.12. *Let A be a C^*-algebra. An element $x \in A$ is said to be positive, if $x = yy^*$ for some $y \in A$.*

Proposition B.13. *Let A be a C^*-algebra and let x be a positive element of A. For each $p > 0$ there exists $y \in A$ such that $x = y^p$.*

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