Frames and a vector-valued ambiguity function

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Outline

1. Problem and goal
2. Frames
3. Multiplication problem and A_p^1
4. $A_p^d : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}^d$, $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$
5. $A_p^d(u)$ for DFT frames
6. Figure
7. Epilogue

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Background

- Originally, our problem was to construct libraries of phase-coded waveforms v parameterized by design variables, for communications and radar.
- A goal was to achieve diverse ambiguity function behavior of v by defining new classes of quadratic phase and number theoretic perfect autocorrelation codes u with which to define v.
- A realistic more general problem was to construct vector-valued waveforms v in terms of vector-valued perfect autocorrelation codes u. Such codes are relevant in light of vector sensor and MIMO capabilities and modeling.
- Example: Discrete time data vector $u(k)$ for a d-element array,

$$k \longmapsto u(k) = (u_0(k), \ldots, u_{d-1}(k)) \in \mathbb{C}^d.$$

We can have $\mathbb{R}^N \to GL(d, \mathbb{C})$, or even more general.

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General problem and STFT theme

- Establish the theory of vector-valued ambiguity functions to estimate v in terms of ambiguity data.
- First, establish this estimation theory by defining the discrete periodic vector-valued ambiguity function in a natural way.
- Mathematically, this natural way is to formulate the discrete periodic vector-valued ambiguity function in terms of the Short Time Fourier Transform (STFT).

STFT and ambiguity function

Short time Fourier transform – STFT

- The narrow band cross-correlation ambiguity function of v, w defined on $\mathbb{R}$ is

$$A(v,w)(t,\gamma) = \int_{\mathbb{R}} v(s+t)\overline{w(s)}e^{-2\pi i s\gamma}\, ds.$$

- $A(v,w)$ is the STFT of v with window w.
- The *narrow band radar ambiguity function* $A(v)$ of v on $\mathbb{R}$ is

$$A(v)(t,\gamma) = \int_{\mathbb{R}} v(s+t)\overline{v(s)}e^{-2\pi i s\gamma}\, ds$$

$$= e^{\pi i t\gamma} \int_{\mathbb{R}} v\left(s+\frac{t}{2}\right) \overline{v\left(s-\frac{t}{2}\right)} e^{-2\pi i s\gamma}\, ds, \ \text{for } (t,\gamma) \in \mathbb{R}^2.$$

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Goal

- Let v be a phase coded waveform with N lags defined by the code u.
- Let u be N-periodic, and so $u : \mathbb{Z}_N \longrightarrow \mathbb{C}$, where $\mathbb{Z}_N$ is the additive group of integers modulo N.
- The *discrete periodic ambiguity function* $A_p(u) : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}$ is

$$A_p(u)(m,n) = \frac{1}{N} \sum_{k=0}^{N-1} u(m+k)\overline{u(k)} e^{-2\pi i k n/N}.$$

Goal

Given a vector valued N-periodic code $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$, construct the following in a meaningful, computable way:

- Generalized $\mathbb{C}$-valued periodic ambiguity function $A_p^1(u) : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}$
- $\mathbb{C}^d$-valued periodic ambiguity function $A_p^d(u) : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}^d$

The STFT is the *guide* and the *theory of frames* is the technology to obtain the goal.

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7. Epilogue

FUNTFs

- A sequence $F = \{E_j\}_{j=1}^{N} \subseteq \mathbb{C}^d$ is a *frame* for $\mathbb{C}^d$ if

$$\exists A, B > 0 \quad \text{such that} \quad \forall u \in \mathbb{C}^d, \quad A\|u\|^2 \leq \sum_{j=1}^{N} |\langle u, E_j\rangle|^2 \leq B\|u\|^2.$$

- F is a *tight frame* if $A = B$; and F is a *finite unit-norm tight frame* (FUNTF) if $A = B$ and each $\|E_j\| = 1$.
- Theorem: If $\{E_j\}_{j=0}^{N-1}$ is a FUNTF for $\mathbb{C}^d$, then

$$\forall u \in \mathbb{C}^d, \quad u = \frac{d}{N} \sum_{j=0}^{N-1} \langle u, E_j\rangle E_j.$$

- Frames give redundant signal representation to compensate for hardware errors, to ensure numerical stability, and to minimize the effects of noise.

Recent applications of FUNTFs

- Robust transmission of data over erasure channels such as the internet [Casazza, Goyal, Kelner, Kovačević]
- Multiple antenna code design for wireless communications [Hochwald, Marzetta,T. Richardson, Sweldens, Urbanke]
- Multiple description coding [Goyal, Heath, Kovačević, Strohmer,Vetterli]
- Quantum detection [Bölcskei, Eldar, Forney, Oppenheim, Kebo, B]
- Grassmannian "min-max" waveforms [Calderbank, Conway, Sloane, et al., Kolesar, B]

DFT FUNTFs

Let $N \geq d$ and form an $N \times d$ matrix using any d columns of the $N \times N$ DFT matrix $(e^{2\pi ijk/N})_{j,k=0}^{N-1}$. The rows of this $N \times d$ matrix, up to multiplication by $\frac{1}{\sqrt{d}}$, form a FUNTF for $\mathbb{C}^d$.

$$N = 8,\, d = 5 \qquad \frac{1}{\sqrt{5}} \begin{bmatrix} * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \end{bmatrix}$$

$$x_m = \frac{1}{5}(e^{2\pi i \frac{m}{8}}, e^{2\pi i m \frac{2}{8}}, e^{2\pi i m \frac{5}{8}}, e^{2\pi i m \frac{6}{8}}, e^{2\pi i m \frac{7}{8}})$$

$$m = 1, \ldots, 8.$$

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7. Epilogue

Multiplication problem

- Given $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$.
- If $d = 1$ and $e_n = e^{2\pi i n/N}$, then

$$A_p(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\langle u(m+k), u(k)e_{nk}\rangle.$$

Multiplication problem

To characterize sequences $\{E_k\} \subseteq \mathbb{C}^d$ and multiplications $*$ so that

$$A_p^1(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\langle u(m+k), u(k) * E_{nk}\rangle \in \mathbb{C}$$

is a meaningful and well-defined *ambiguity function.* This formula is clearly motivated by the STFT.

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Ambiguity function assumptions

There is a natural way to address the multiplication problem motivated by the fact that $e_m e_n = e_{m+n}$. To this end, we shall make the *ambiguity function assumptions*:

- $\exists \, \{E_k\}_{k=0}^{N-1} \subseteq \mathbb{C}^d$ and a multiplication $*$ such that $E_m * E_n = E_{m+n}$ for $m, n \in \mathbb{Z}_N$;
- $\{E_k\}_{k=0}^{N-1} \subseteq \mathbb{C}^d$ is a tight frame for $\mathbb{C}^d$;
- $* : \mathbb{C}^d \times \mathbb{C}^d \longrightarrow \mathbb{C}^d$ is bilinear, in particular,

$$\left(\sum_{j=0}^{N-1} c_j E_j\right) * \left(\sum_{k=0}^{N-1} d_k E_k\right) = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} c_j d_k E_j * E_k.$$

Calculation

- Let $\{E_j\}_j^{N-1} \subseteq \mathbb{C}^d$ satisfy the three ambiguity function assumptions.
- Given $u, v : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$ and $m, n \in \mathbb{Z}_N$.
- Then, one calculates

$$u(m) * v(n) \quad = \quad \frac{d^2}{N^2} \sum_{j=0}^{N-1} \sum_{s=0}^{N-1} \langle u(m), E_j \rangle \langle v(n), E_s \rangle E_{j+s}.$$

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$A_p^1(u)$ for DFT frames

- Let $\{E_j\}_j^{N-1} \subseteq \mathbb{C}^d$ satisfy the three ambiguity function assumptions.
- Further, assume that $\{E_j\}_{j=0}^{N-1}$ is a DFT frame, and let r designate a fixed column.
- Without loss of generality, choose the first d columns of the $N \times N$ DFT matrix.
- Then, one calculates

$$\begin{aligned} E_m * E_n(r) &= \frac{d^2}{N^2} \sum_{j=0}^{N-1} \sum_{s=0}^{N-1} \langle E_m, E_j \rangle \langle E_n, E_s \rangle E_{j+s}(r). \\ &= \frac{e_{(m+n)r}}{\sqrt{d}} = E_{m+n}(r). \end{aligned}$$

$A_p^1(u)$ for DFT frames

- Thus, for DFT frames, $*$ is componentwise multiplication in $\mathbb{C}^d$ with a factor of $\sqrt{d}$.
- In this case $A_p^1(u)$ is well-defined for $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$ by

$$
\begin{aligned}
A_p^1(u)(m,n) &= \frac{1}{N}\sum_{k=0}^{N-1} \langle u(m+k), u(k) * E_{nk} \rangle \\
&= \frac{d}{N^2}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1} \langle E_j, u(k) \rangle \langle u(m+k), E_{j+nk} \rangle .
\end{aligned}
$$

Remark

- In the previous DFT example, $*$ is intrinsically related to the "addition" defined on the indices of the frame elements, viz., $E_m * E_n = E_{m+n}$.
- Alternatively, we could have $E_m * E_n = E_{m \bullet n}$ for some function $\bullet : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{Z}_N$, and, thereby, we could use frames which are not FUNTFs.
- Given a bilinear multiplication $* : \mathbb{C}^d \times \mathbb{C}^d \longrightarrow \mathbb{C}^d$, we can find a frame $\{E_j\}_j$ and an index operation $\bullet$ with the $E_m * E_n = E_{m \bullet n}$ property.
- If $\bullet$ is the multiplication for a group, possibly non-abelian and/or infinite, we may reverse the process and find a FUNTF and bilinear multiplication $*$ with the $E_m * E_n = E_{m \bullet n}$ property.

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$A_p^1(u)$ for cross product frames

- Take $*: \mathbb{C}^3 \times \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ to be the cross product on $\mathbb{C}^3$ and let $\{i, j, k\}$ be the standard basis.
- $i * j = k$, $j * i = -k$, $k * i = j$, $i * k = -j$, $j * k = i$, $k * j = -i$, $i * i = j * j = k * k = 0$. $\{0, i, j, k, -i, -j, -k, \}$ is a tight frame for $\mathbb{C}^3$ with frame constant 2. Let $E_0 = 0, E_1 = i, E_2 = j, E_3 = k, E_4 = -i, E_5 = -j, E_6 = -k$.
- The index operation corresponding to the frame multiplication is the non-abelian operation $\bullet : \mathbb{Z}_7 \times \mathbb{Z}_7 \longrightarrow \mathbb{Z}_7$, where $1 \bullet 2 = 3, 2 \bullet 1 = 6, 3 \bullet 1 = 2, 1 \bullet 3 = 5, 2 \bullet 3 = 1, 3 \bullet 2 = 4, 1 \bullet 1 = 2 \bullet 2 = 3 \bullet 3 = 0, n \bullet 0 = 0 \bullet n = 0, 1 \bullet 4 = 0, 1 \bullet 5 = 6, 1 \bullet 6 = 2, 4 \bullet 1 = 0, 5 \bullet 1 = 3, 6 \bullet 1 = 5, 2 \bullet 4 = 3, 2 \bullet 5 = 0$, etc.
- The three ambiguity function assumptions are valid and so we can write the cross product as

$$u \times v = u * v = \frac{1}{2^2} \sum_{s=1}^{6} \sum_{t=1}^{6} \langle u, E_s \rangle \langle v, E_t \rangle E_{s \bullet t}.$$

- Consequently, $A_p^1(u)$ can be well-defined.

Vector-valued ambiguity function $A_p^d(u)$

- Let $\{E_j\}_j^{N-1} \subseteq \mathbb{C}^d$ satisfy the three ambiguity function assumptions.
- Given $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$.
- The following definition is clearly *motivated* by the STFT.

Definition

$A_p^d(u) : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}^d$ is defined by

$$A_p^d(u)(m,n) = \frac{1}{N} \sum_{k=0}^{N-1} u(m+k) * \overline{u(k)} * \overline{E_{nk}}.$$

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STFT formulation of $A_p(u)$

- The discrete periodic ambiguity function of $u : \mathbb{Z}_N \longrightarrow \mathbb{C}$ can be written as

$$A_p(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1} \langle \tau_m u(k), F^{-1}(\tau_n \hat{u})(k)\rangle,$$

  where $\tau_(m)u(k) = u(m+k)$ is translation by m and $F^{-1}(u)(k)) = \check{u}(k)$ is Fourier inversion.
- As such we see that $A_p(u)$ has the form of a STFT.
- We shall develop a vector-valued DFT theory to *verify* (not just *motivate*) that $A_p^d(u)$ is an STFT in the case $\{E_k\}_{k=0}^{N-1}$ is a DFT frame for $\mathbb{C}^d$.

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DFT frames and the vector-valued DFT

Definition

Given $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$, and let $\{E_k\}_{k=0}^{N-1}$ be a DFT frame for $\mathbb{C}^d$. The *vector-valued discrete Fourier transform* of u is

$$\forall\ n \in \mathbb{Z}_N, \quad F(u)(n) = \hat{u}(n) = \sum_{m=0}^{N-1} u(m) * E_{mn},$$

where $*$ is pointwise (coordinatewise) multiplication.

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Vector-valued Fourier inversion theorem

- Inversion process for the vector-valued case is analogous to the 1-dimensional case.
- We must define a new multiplication in the frequency domain to avoid divisibility problems.
- Define the weighted multiplication $(*) : \mathbb{C}^d \times \mathbb{C}^d \longrightarrow \mathbb{C}^d$ by $u(*)v = u * v * \omega$ where $\omega = (\omega_1, \ldots, \omega_d)$ has the property that each $\omega_n = \frac{1}{\#\{m \in \mathbb{Z}_N : mn = 0\}}$.
- For the following theorem assume $d << N$ or N prime.

Theorem - Vector-valued Fourier inversion

The vector valued Fourier transform F is an isomorphism from $\ell^2(\mathbb{Z}_N)$ to $\ell^2(\mathbb{Z}_N, \omega)$ with inverse

$$\forall\ m \in \mathbb{Z}_N, \quad F^{-1}(m) = u(m) = \frac{d}{N} \sum_{n=0}^{N-1} \hat{u}(n) * E_{-mn} * \omega.$$

N prime implies F is unitary.

$A_p^d(u)$ as an STFT

- Given $u, v : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$, and let $\{E_k\}_{k=0}^{N-1}$ be a DFT frame for $\mathbb{C}^d$.
- $u * \overline{v}$ denotes pointwise (coordinatewise) multiplication with a factor of $\sqrt{d}$.
- We compute

$$A_p^d(u)(m,n) = \frac{1}{N} \sum_{k=0}^{N-1} (\tau_m u(k)) * \overline{F^{-1}(\tau_n \hat{u})(k)}.$$

- Thus, $A_p^d(u)$ is compatible with point of view of defining a vector-valued ambiguity function in the context of the STFT.

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6. Figure
7. Epilogue

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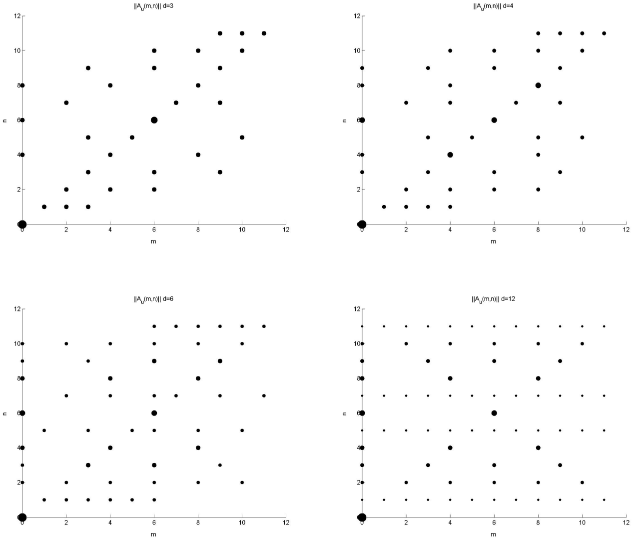
Intensity $||A_u^d(m,n)||$ for $d = 3, 4, 6, 12$ where $u : \mathbb{Z}_N \to \mathbb{C}^d$ is a Wiener CAZAC



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6. Figure
7. Epilogue

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Epilogue

- If $(G, \bullet)$ is a finite group with representation $\rho : G \longrightarrow GL(\mathbb{C}^d)$, then there is a frame $\{E_n\}_{n \in G}$ and bilinear multiplication, $* : \mathbb{C}^d \times \mathbb{C}^d \longrightarrow \mathbb{C}^d$, such that $E_m * E_n = E_{m \bullet n}$. Thus, we can develop $A_p^d(u)$ theory in this setting.
- Analyze ambiguity function behavior for (phase-coded) vector-valued waveforms $v : \mathbb{R} \longrightarrow \mathbb{C}^d$, defined by $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$ as

$$v = \sum_{k=0}^{N-1} u(k) \mathbb{1}_{[kT,(k+1)T)},$$

in terms of $A_p^d(u)$. (See Figure)

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That's all folks!

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