Frames and a vector-valued ambiguity function

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Originally, our problem was to construct libraries of phase-coded waveforms $\nu$ parameterized by design variables, for communications and radar.

A goal was to achieve diverse ambiguity function behavior of $\nu$ by defining new classes of quadratic phase and number theoretic perfect autocorrelation codes $u$ with which to define $\nu$.

A realistic more general problem was to construct vector-valued waveforms $\nu$ in terms of vector-valued perfect autocorrelation codes $u$. Such codes are relevant in light of vector sensor and MIMO capabilities and modeling.

Example: Discrete time data vector $u(k)$ for a $d$-element array,

\[ k \mapsto u(k) = (u_0(k), \ldots, u_{d-1}(k)) \in \mathbb{C}^d. \]

We can have $\mathbb{R}^N \to GL(d, \mathbb{C})$, or even more general.
Establish the theory of vector-valued ambiguity functions to estimate $\nu$ in terms of ambiguity data.

First, establish this estimation theory by defining the discrete periodic vector-valued ambiguity function in a natural way.

Mathematically, this natural way is to formulate the discrete periodic vector-valued ambiguity function in terms of the Short Time Fourier Transform (STFT).
The narrow band cross-correlation ambiguity function of $v, w$ defined on $\mathbb{R}$ is

$$A(v, w)(t, \gamma) = \int_{\mathbb{R}} v(s + t)\overline{w(s)}e^{-2\pi is\gamma}ds.$$ 

$A(v, w)$ is the STFT of $v$ with window $w$.

The narrow band radar ambiguity function $A(v)$ of $v$ on $\mathbb{R}$ is

$$A(v)(t, \gamma) = \int_{\mathbb{R}} v(s + t)v(s)e^{-2\pi is\gamma}ds$$

$$= e^{\pi it\gamma} \int_{\mathbb{R}} v \left( s + \frac{t}{2} \right)\overline{v \left( s - \frac{t}{2} \right)}e^{-2\pi is\gamma}ds, \text{ for } (t, \gamma) \in \mathbb{R}^2.$$
Goal

Let \( \nu \) be a phase coded waveform with \( N \) lags defined by the code \( u \).

Let \( u \) be \( N \)-periodic, and so \( u : \mathbb{Z}_N \rightarrow \mathbb{C} \), where \( \mathbb{Z}_N \) is the additive group of integers modulo \( N \).

The discrete periodic ambiguity function \( A_p(u) : \mathbb{Z}_N \times \mathbb{Z}_N \rightarrow \mathbb{C} \) is

\[
A_p(u)(m, n) = \frac{1}{N} \sum_{k=0}^{N-1} u(m + k) \overline{u(k)} e^{-2\pi i kn/N}.
\]

Goal

Given a vector valued \( N \)-periodic code \( u : \mathbb{Z}_N \rightarrow \mathbb{C}^d \), construct the following in a meaningful, computable way:

- Generalized \( \mathbb{C} \)-valued periodic ambiguity function \( A^1_p(u) : \mathbb{Z}_N \times \mathbb{Z}_N \rightarrow \mathbb{C} \)
- \( \mathbb{C}^d \)-valued periodic ambiguity function \( A^d_p(u) : \mathbb{Z}_N \times \mathbb{Z}_N \rightarrow \mathbb{C}^d \)

The STFT is the \textit{guide} and the \textit{theory of frames} is the technology to obtain the goal.
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FUNTFs

- A sequence $F = \{E_j\}_{j=1}^N \subseteq \mathbb{C}^d$ is a frame for $\mathbb{C}^d$ if

$$\exists A, B > 0 \text{ such that } \forall u \in \mathbb{C}^d, \quad A \|u\|^2 \leq \sum_{j=1}^N |\langle u, E_j \rangle|^2 \leq B \|u\|^2.$$  

- $F$ is a tight frame if $A = B$; and $F$ is a finite unit-norm tight frame (FUNTF) if $A = B$ and each $\|E_j\| = 1$.

- Theorem: If $\{E_j\}_{j=0}^{N-1}$ is a FUNTF for $\mathbb{C}^d$, then

$$\forall u \in \mathbb{C}^d, \quad u = \frac{d}{N} \sum_{j=0}^{N-1} \langle u, E_j \rangle E_j.$$  

- Frames give redundant signal representation to compensate for hardware errors, to ensure numerical stability, and to minimize the effects of noise.
Recent applications of FUNTFs

- Robust transmission of data over erasure channels such as the internet [Casazza, Goyal, Kelner, Kovačević]
- Multiple antenna code design for wireless communications [Hochwald, Marzetta, T. Richardson, Sweldens, Urbanke]
- Multiple description coding [Goyal, Heath, Kovačević, Strohmer, Vetterli]
- Quantum detection [Bölcskei, Eldar, Forney, Oppenheim, Kebo, B]
- Grassmannian "min-max" waveforms [Calderbank, Conway, Sloane, et al., Kolesar, B]
Let $N \geq d$ and form an $N \times d$ matrix using any $d$ columns of the $N \times N$ DFT matrix $(e^{2\pi i jk/N})_{j,k=0}^{N-1}$. The rows of this $N \times d$ matrix, up to multiplication by $\frac{1}{\sqrt{d}}$, form a FUNTF for $\mathbb{C}^d$.

$$ \begin{bmatrix} * & * & \cdots & * & * & * \\ * & * & \cdots & * & * & * \\ * & * & \cdots & * & * & * \\ * & * & \cdots & * & * & * \\ * & * & \cdots & * & * & * \\ * & * & \cdots & * & * & * \end{bmatrix} $$

$$x_m = \frac{1}{5}(e^{2\pi i m/8}, e^{2\pi i m^2/8}, e^{2\pi i m^5/8}, e^{2\pi i m^6/8}, e^{2\pi i m^7/8})$$

$m = 1, \ldots, 8$. 
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Given \( u : \mathbb{Z}_N \rightarrow \mathbb{C}^d \).

If \( d = 1 \) and \( e_n = e^{2\pi in/N} \), then

\[
A_p(u)(m, n) = \frac{1}{N} \sum_{k=0}^{N-1} \langle u(m + k), u(k)e_{nk} \rangle.
\]

To characterize sequences \( \{E_k\} \subseteq \mathbb{C}^d \) and multiplications \( \ast \) so that

\[
A^1_p(u)(m, n) = \frac{1}{N} \sum_{k=0}^{N-1} \langle u(m + k), u(k) \ast E_{nk} \rangle \in \mathbb{C}
\]

is a meaningful and well-defined *ambiguity function*. This formula is clearly motivated by the STFT.
There is a natural way to address the multiplication problem motivated by the fact that $e_m e_n = e_{m+n}$. To this end, we shall make the *ambiguity function assumptions*:

- $\exists \{E_k\}_{k=0}^{N-1} \subseteq \mathbb{C}^d$ and a multiplication $\ast$ such that $E_m \ast E_n = E_{m+n}$ for $m, n \in \mathbb{Z}_N$;
- $\{E_k\}_{k=0}^{N-1} \subseteq \mathbb{C}^d$ is a tight frame for $\mathbb{C}^d$;
- $\ast : \mathbb{C}^d \times \mathbb{C}^d \longrightarrow \mathbb{C}^d$ is bilinear, in particular,

$$
\left( \sum_{j=0}^{N-1} c_j E_j \right) \ast \left( \sum_{k=0}^{N-1} d_k E_k \right) = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} c_j d_k E_j \ast E_k.
$$
Let \( \{E_j\}_{j=0}^{N-1} \subseteq \mathbb{C}^d \) satisfy the three ambiguity function assumptions.

Given \( u, v : \mathbb{Z}_N \longrightarrow \mathbb{C}^d \) and \( m, n \in \mathbb{Z}_N \).

Then, one calculates

\[
  u(m) \ast v(n) = \frac{d^2}{N^2} \sum_{j=0}^{N-1} \sum_{s=0}^{N-1} \langle u(m), E_j \rangle \langle v(n), E_s \rangle E_{j+s}.
\]
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Let \( \{ E_j \}_{j=0}^{N-1} \subseteq \mathbb{C}^d \) satisfy the three ambiguity function assumptions.

Further, assume that \( \{ E_j \}_{j=0}^{N-1} \) is a DFT frame, and let \( r \) designate a fixed column.

Without loss of generality, choose the first \( d \) columns of the \( N \times N \) DFT matrix.

Then, one calculates

\[
E_m \ast E_n(r) = \frac{d^2}{N^2} \sum_{j=0}^{N-1} \sum_{s=0}^{N-1} \langle E_m, E_j \rangle \langle E_n, E_s \rangle E_{j+s}(r) = \frac{e^{i(m+n)r}}{\sqrt{d}} = E_{m+n}(r).
\]
Thus, for DFT frames, \( \star \) is componentwise multiplication in \( \mathbb{C}^d \) with a factor of \( \sqrt{d} \).

In this case \( A_p^1(u) \) is well-defined for \( u : \mathbb{Z}_N \rightarrow \mathbb{C}^d \) by

\[
A_p^1(u)(m, n) = \frac{1}{N} \sum_{k=0}^{N-1} \langle u(m + k), u(k) \star E_{nk} \rangle
\]

\[
= \frac{d}{N^2} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \langle E_j, u(k) \rangle \langle u(m + k), E_{j+nk} \rangle.
\]
In the previous DFT example, $*$ is intrinsically related to the “addition” defined on the indices of the frame elements, viz., $E_m * E_n = E_{m+n}$.

Alternatively, we could have $E_m * E_n = E_{m \cdot n}$ for some function $\cdot : \mathbb{Z}_N \times \mathbb{Z}_N \rightarrow \mathbb{Z}_N$, and, thereby, we could use frames which are not FUNTFs.

Given a bilinear multiplication $* : \mathbb{C}^d \times \mathbb{C}^d \rightarrow \mathbb{C}^d$, we can find a frame $\{E_j\}_j$ and an index operation $\cdot$ with the $E_m * E_n = E_{m \cdot n}$ property.

If $\cdot$ is the multiplication for a group, possibly non-abelian and/or infinite, we may reverse the process and find a FUNTF and bilinear multiplication $*$ with the $E_m * E_n = E_{m \cdot n}$ property.
Take $\star : \mathbb{C}^3 \times \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ to be the cross product on $\mathbb{C}^3$ and let $\{i, j, k\}$ be the standard basis.

$i \star j = k, j \star i = -k, k \star i = j, i \star k = -j, j \star k = i, k \star j = -i,$

$i \star i = j \star j = k \star k = 0$. $\{0, i, j, k, -i, -j, -k\}$ is a tight frame for $\mathbb{C}^3$ with frame constant 2. Let $E_0 = 0, E_1 = i, E_2 = j, E_3 = k, E_4 = -i, E_5 = -j, E_6 = -k$.

The index operation corresponding to the frame multiplication is the non-abelian operation $\bullet : \mathbb{Z}_7 \times \mathbb{Z}_7 \longrightarrow \mathbb{Z}_7$, where

$1 \bullet 2 = 3, 2 \bullet 1 = 6, 3 \bullet 1 = 2, 1 \bullet 3 = 5, 2 \bullet 3 = 1, 3 \bullet 2 = 4, 1 \bullet 1 = 2, 2 \bullet 2 = 3, 3 \bullet 0 = 0, n \bullet 0 = 0, n \bullet n = 0, 1 \bullet 4 = 0, 1 \bullet 5 = 6, 1 \bullet 6 = 2, 4 \bullet 1 = 0, 5 \bullet 1 = 3, 6 \bullet 1 = 5, 2 \bullet 4 = 3, 2 \bullet 5 = 0$, etc.

The three ambiguity function assumptions are valid and so we can write the cross product as

$$u \times v = u \star v = \frac{1}{2^2} \sum_{s=1}^{6} \sum_{t=1}^{6} \langle u, E_s \rangle \langle v, E_t \rangle E_{s \bullet t}. $$

Consequently, $A_{\rho}^1(u)$ can be well-defined.
Let \( \{ E_j \}_{j=0}^{N-1} \subseteq \mathbb{C}^d \) satisfy the three ambiguity function assumptions.

Given \( u : \mathbb{Z}_N \rightarrow \mathbb{C}^d \).

The following definition is clearly motivated by the STFT.

**Definition**

\[
A_p^d(u) : \mathbb{Z}_N \times \mathbb{Z}_N \rightarrow \mathbb{C}^d
\]

is defined by

\[
A_p^d(u)(m, n) = \frac{1}{N} \sum_{k=0}^{N-1} u(m + k) \ast \overline{u(k)} \ast \overline{E_{nk}}.
\]
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The discrete periodic ambiguity function of $u : \mathbb{Z}_N \rightarrow \mathbb{C}$ can be written as

$$A_p(u)(m, n) = \frac{1}{N} \sum_{k=0}^{N-1} \langle \tau_m u(k), F^{-1}(\tau_n \hat{u})(k) \rangle,$$

where $\tau_m u(k) = u(m + k)$ is translation by $m$ and $F^{-1}(u)(k) = \hat{u}(k)$ is Fourier inversion.

As such we see that $A_p(u)$ has the form of a STFT.

We shall develop a vector-valued DFT theory to verify (not just motivate) that $A_p^d(u)$ is an STFT in the case $\{E_k\}_{k=0}^{N-1}$ is a DFT frame for $\mathbb{C}^d$. 
Definition

Given \( u : \mathbb{Z}_N \rightarrow \mathbb{C}^d \), and let \( \{ E_k \}_{k=0}^{N-1} \) be a DFT frame for \( \mathbb{C}^d \). The vector-valued discrete Fourier transform of \( u \) is

\[
\forall \ n \in \mathbb{Z}_N, \quad F(u)(n) = \hat{u}(n) = \sum_{m=0}^{N-1} u(m) \ast E_{mn},
\]

where \( \ast \) is pointwise (coordinatewise) multiplication.
Inversion process for the vector-valued case is analogous to the 1-dimensional case.

We must define a new multiplication in the frequency domain to avoid divisibility problems.

Define the weighted multiplication \((\ast) : \mathbb{C}^d \times \mathbb{C}^d \rightarrow \mathbb{C}^d\) by
\[
(\mathbf{u}\ast \mathbf{v}) = \mathbf{u} \ast \mathbf{v} \ast \mathbf{\omega}
\]
where \(\mathbf{\omega} = (\omega_1, \ldots, \omega_d)\) has the property that each
\[
\omega_n = \frac{1}{\#\{ m \in \mathbb{Z}_N : mn = 0 \}}.
\]

For the following theorem assume \(d << N\) or \(N\) prime.

**Theorem - Vector-valued Fourier inversion**

The vector valued Fourier transform \(F\) is an isomorphism from \(\ell^2(\mathbb{Z}_N)\) to \(\ell^2(\mathbb{Z}_N, \omega)\) with inverse

\[
\forall \ m \in \mathbb{Z}_N, \quad F^{-1}(m) = u(m) = \frac{d}{N} \sum_{n=0}^{N-1} \hat{u}(n) \ast E_{-mn} \ast \omega.
\]

\(N\) prime implies \(F\) is unitary.
$A^d_p(u)$ as an STFT

- Given $u, v : \mathbb{Z}_N \rightarrow \mathbb{C}^d$, and let $\{E_k\}_{k=0}^{N-1}$ be a DFT frame for $\mathbb{C}^d$.
- $u \ast \overline{v}$ denotes pointwise (coordinatewise) multiplication with a factor of $\sqrt{d}$.
- We compute

$$A^d_p(u)(m, n) = \frac{1}{N} \sum_{k=0}^{N-1} (\tau_m u(k)) \ast F^{-1}(\tau_n \hat{u})(k).$$

- Thus, $A^d_p(u)$ is compatible with point of view of defining a vector-valued ambiguity function in the context of the STFT.
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Intensity $\|A_u^d(m, n)\|$ for $d = 3, 4, 6, 12$ where $u : \mathbb{Z}_N \rightarrow \mathbb{C}^d$ is a Wiener CAZAC.
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If \((G, \cdot)\) is a finite group with representation \(\rho : G \rightarrow GL(\mathbb{C}^d)\), then there is a frame \(\{E_n\}_{n \in G}\) and bilinear multiplication, \(* : \mathbb{C}^d \times \mathbb{C}^d \rightarrow \mathbb{C}^d\), such that \(E_m \ast E_n = E_{m\cdot n}\). Thus, we can develop \(A^d_p(u)\) theory in this setting.

Analyze ambiguity function behavior for (phase-coded) vector-valued waveforms \(v : \mathbb{R} \rightarrow \mathbb{C}^d\), defined by \(u : \mathbb{Z}_N \rightarrow \mathbb{C}^d\) as

\[
v = \sum_{k=0}^{N-1} u(k) \mathbb{1}_{[kT,(k+1)T)},\]

in terms of \(A^d_p(u)\). (See Figure)
That's all folks!