Balayage and the theory of generalized Fourier frames

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Fourier frames, goal, and a litany of names

Definition

E $= \{x_n\} \subseteq \mathbb{R}^d, \Lambda \subseteq \widehat{\mathbb{R}}^d$. E is a *Fourier frame* for $L^2(\Lambda)$ if

$\exists A, B > 0, \forall F \in L^2(\Lambda)$,

$$A \; ||F||^2_{L^2(\Lambda)} \leq \sum_n |< F(\gamma), e^{-2\pi i x_n \cdot \gamma} > |^2 \leq B \; ||F||^2_{L^2(\Lambda)}.$$

- *Goal* Formulate a general theory of Fourier frames and non-uniform sampling formulas parametrized by the space $M(\mathbb{R}^d)$ of bounded Radon measures.
- *Motivation* Beurling theory (1959-1960).
- *Names* Riemann-Weber, Dini, G.D. Birkhoff, Paley-Wiener, Levinson, Duffin-Schaeffer, Beurling-Malliavin, Beurling, H.J. Landau, Jaffard, Seip, Ortega-Certà–Seip.

Balayage

- Let $M(G)$ be the algebra of bounded Radon measures on the LCAG G.
- Balayage in potential theory was introduced by Christoffel (early 1870s) and Poincaré (1890).

Definition

(Beurling) Balayage is possible for $(E, \Lambda) \subseteq G \times \widehat{G}$, a LCAG pair, if

$$\forall \mu \in M(G), \exists \nu \in M(E) \text{ such that } \hat{\mu} = \hat{\nu} \text{ on } \Lambda.$$

We write balayage (E, Λ).

- The set, Λ, of group characters is the analogue of the original role of Λ in balayage as a collection of potential theoretic kernels.
- Kahane formulated balayage for the harmonic analysis of restriction algebras.

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Spectral synthesis

Definition

(Wiener, Beurling) Closed $\Lambda \subseteq \widehat{G}$ is a set of *spectral synthesis* (S-set) if

$$\forall \mu \in M(G), \forall f \in C_b(G),$$
$$\text{supp}(\widehat{f}) \subseteq \Lambda \text{ and } \hat{\mu} = 0 \text{ on } \Lambda \Longrightarrow \int_G f \, d\mu = 0.$$

$(\forall T \in A'(\widehat{G}), \forall \phi \in A(\widehat{G}), \quad \text{supp}(T) \subseteq \Lambda \text{ and } \phi = 0 \text{ on } \Lambda \Rightarrow T(\phi) = 0.)$

- Ideal structure of $L^1(G)$ - the Nullstellensatz of harmonic analysis
- $T \in D'(\widehat{\mathbb{R}}^d), \phi \in C_c^\infty(\widehat{\mathbb{R}}^d)$, and $\phi = 0$ on $\text{supp}(T) \Rightarrow T(\phi) = 0$, with same result for $M(\widehat{\mathbb{R}}^d)$ and $C_0(\widehat{\mathbb{R}}^d)$.
- $S^2 \subseteq \widehat{\mathbb{R}}^3$ is not an S-set (L. Schwartz), and every non-discrete $\widehat{G}$ has non-S-sets (Malliavin).
- Polyhedra are S-sets. The $\frac{1}{3}$-Cantor set is an S-set with non-S-subsets.

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Strict multiplicity

Definition

$\Gamma \subseteq \widehat{G}$ is a set of *strict multiplicity* if

$$\exists\ \mu \in M(\Gamma)\backslash\{0\} \text{ such that } \check{\mu} \text{ vanishes at infinity in } G.$$

- Riemann and sets of uniqueness in the wide sense.
- Menchov (1916): $\exists$ closed $\Gamma \subseteq \widehat{\mathbb{R}/\mathbb{Z}}$ and $\mu \in M(\Gamma)\backslash\{0\}$,
$$|\Gamma| = 0 \text{ and } \check{\mu}(n) = O((\log|n|)^{-1/2}), |n| \to \infty.$$
- 20th century history to study rate of decrease: Bary (1927), Littlewood (1936), Salem (1942, 1950), Ivašev-Mucatov (1957), Beurling.

Assumption

$\forall\ \gamma \in \Lambda$ and $\forall\ N(\gamma)$, compact neighborhood, $\Lambda \cap N(\gamma)$ is a set of strict multiplicity.

A theorem of Beurling

Definition

E $= \{x_n\} \subseteq \mathbb{R}^d$ is *separated* if

$$\exists\, r > 0,\ \forall m, n,\ m \neq n \Rightarrow ||x_m - x_n|| \geq r.$$

Theorem

Let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be a compact S-set, symmetric about $0 \in \widehat{\mathbb{R}}^d$, and let E $\subseteq \mathbb{R}^d$ be separated. If balayage (E,Λ), then

$$\textit{E is a Fourier frame for } L^2(\Lambda).$$

- Equivalent formulation in terms of

$$PW_\Lambda = \{f \in L^2(\mathbb{R}^d) : \mathrm{supp}(\hat{f}) \subseteq \Lambda\}.$$

- $\forall F \in L^2(\Lambda), \qquad F = \sum_{x \in \mathsf{E}} < F, S^{-1}(e_x) >_\Lambda e_x$ in $L^2(\Lambda)$.
- For $\mathbb{R}^d$ and other generality beyond Beurling's theorem in $\mathbb{R}$, the result above was formulated by Hui-Chuan Wu and JB (1998), see Landau (1967).

Lower frame bounds

- Let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be a compact S-set, and assume balayage (E, Λ) where E $= \{x_n\}$ is separated.

1. $\forall F \in L^2(\Lambda),\ \Lambda$ convex,
$$\sqrt{A}\ \frac{\int_\Lambda |F(\gamma)+F(2\gamma)+F(3\gamma)|^2\ d\gamma}{(\int_\Lambda |F(\gamma)|^2\ d\gamma)^{1/2}}$$
$$\leq (\sum |\check{F}(x_n)|^2)^{1/2} + \tfrac{1}{2}(\sum |\check{F}(\tfrac{1}{2}x_n)|^2)^{1/2} + \tfrac{1}{3}(\sum |\check{F}(\tfrac{1}{3}x_n)|^2)^{1/2}.$$

2. Given positive $G \in L^2(\Lambda)$. Then $\forall F \in L^2(\Lambda)$,
$$\sqrt{A}\ \frac{\int_\Lambda |F(\gamma)|^2 G(\gamma)\ d\gamma}{(\int_\Lambda |F(\gamma)|^2\ d\gamma)^{\frac{1}{2}}} \leq (\sum |(FG)\check{}(x_n)|^2)^{1/2}.$$

Semi-discrete Gabor frames

Let $G \in L^2(\widehat{\mathbb{R}}^d)$ satisfy $||G||_{L^2(\widehat{\mathbb{R}}^d)} = 1$; let $\Lambda \subset \widehat{\mathbb{R}}^d$ be an S-set, symmetric about 0; and let $E \subset \mathbb{R}^d$ be separated. Define

(STFT) $\quad \forall F \in L^2(\Lambda), \quad V_G F(x,\gamma) = \int_\Lambda F(\lambda) G(\lambda - \gamma) e^{2\pi i x \cdot \lambda} \, d\lambda.$

Theorem

If balayage (E,Λ), then

$$\exists A, B > 0, \quad \forall F \in L^2(\Lambda),$$

$$A \, ||F||^2_{L^2(\Lambda)} \leq \int_{\widehat{\mathbb{R}}^d} \sum_{x \in E} |V_G F(x,\gamma)|^2 \, d\gamma \leq B \, ||F||^2_{L^2(\Lambda)}.$$

Remark There are basic problems to be resolved and there have been fundamental recent advances.

Examples of balayage

1. Let E $\subseteq \mathbb{R}^d$ be separated. Define

$$r = r(E) = \sup_{x \in \mathbb{R}^d} \text{dist}(x, E).$$

   If $r\rho < \frac{1}{4}$, then balayage (E, $\bar{B}(0,\rho)$). $\frac{1}{4}$ is the best possible.

2. If balayage (E, Λ) and $\Lambda_0 \subseteq \Lambda$, then balayage (E, Λ_0).
3. Let E $= \{x_n\}$ be a Fourier frame for PW_Λ. Then for all $\Lambda_0 \subseteq \Lambda$ with $\text{dist}(\Lambda_0, \Lambda^c) > 0$, we have balayage (E, Λ_0).
4. In $\mathbb{R}^1$, for a separated set E, Beurling lower density $> \rho$ is necessary and sufficient for balayage (E, $[\frac{-\rho}{2}, \frac{\rho}{2}]$).

Remark In $\mathbb{R}^1$, if E is uniformly dense in the sense of Duffin-Schaeffer, then $D^-(E), D^+(E)$, and $D_u(E)$ coincide.
So Beurling's result $\Rightarrow$ Duffin-Schaeffer's result on Fourier frames.

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Sampling formulas (1)

- Let $\Lambda \in \widehat{\mathbb{R}}^d$ be a compact S-set, and assume balayage (E, Λ), E $= \{x_n\} \subseteq \mathbb{R}^d$ separated.
- *Theorem* $\exists \epsilon > 0$, balayage (E, Λ_ϵ).
- *Theorem* $\forall x \in \mathbb{R}^d,\ \exists\ \{b_n(x)\} \in l^1(\mathbb{Z})$,

$$\sup_{x\in\mathbb{R}^d} \sum_n |b_n(x)| \leq K(E, \Lambda_\epsilon)$$

and $e^{-2\pi i x\cdot\gamma} = \sum_n b_n(x) e^{-2\pi i x_n\cdot\gamma}$ uniformly on Λ_ϵ.

- Let h be entire on $\mathbb{R}^d$ with $e^{-\Omega(|x|)}$ decay,

$$h(0) = 1 \text{ and } \mathrm{supp}(\hat{h}) \subseteq \bar{B}(0, \epsilon).$$

Theorem

$\forall f \in C_b(\mathbb{R}), supp(\hat{f}) \subseteq \Lambda,$

$$\forall y \in \mathbb{R}^d, \quad f(y) = \sum f(x_n) b_n(y) h(x_n - y)$$

- Weighted sampling function $b_n(y)h(x_n - y)$ independent of $f \in C_b(\mathbb{R}^d), \mathrm{supp}(\hat{f}) \subseteq \Lambda$.

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Sampling formulas (2)

- The Nyquist condition, $2T\Omega \leq 1$, for sampling period T and bandwidth $[-\Omega, \Omega]$, gives way to balayage (E,Λ), where Λ is the bandwidth and the sampling set E is related to Λ by balayage (E,Λ).
- Let $s \in C_b(\mathbb{R}^d)$, supp$(\hat{s}) \subseteq \Lambda$, a compact S-set - *sampling function* s.
- Let $A = \{a(n)\} \subseteq \mathbb{R}^d, n \in \mathbb{Z}$ and distinct points $a(n)$. Define

$$V_A = \{f \in C_b(\mathbb{R}^d) : \forall x \in \mathbb{R}^d, f(x) = \sum_n c_n(f)s(x-a(n)), \sum_n |c_n(f)| < \infty\}.$$

- Assume balayage (E,Λ), E $= \{x_n\} \subseteq \mathbb{R}^d$ separated.
- Define

$$V_E = \{f \in C_b(\mathbb{R}^d) : \forall x \in \mathbb{R}^d, f(x) = \sum_n c_n(f)s(x-x_n), \sum_n |c_n(f)| < \infty\}.$$

Theorem

$V = \cup_A V_A \subseteq V_E \subseteq C_b(\mathbb{R}^d)$. *Thus,*

$$\forall f \in V, \quad f(x) = \textstyle\sum_n c_n(f)s(x - x_n), \text{ uniformly on } \mathbb{R}^d.$$

The Theory

- Signal decomposition in terms of (E, Λ)-balayage, defined by measures whose absolutely convergent non-harmonic Fourier series are generalized characters parameterized by Λ.
- Sampling multipliers and lower frame bound inequalities
- Pseudo-differential operator sampling formulas
- Bilinear frame operators and classical extensions of the Calderon formula in harmonic analysis

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That's all folks!