# Frames: CAZAC sequences and potential theory for sampling and classification

John J. Benedetto

Norbert Wiener Center
Department of Mathematics
University of Maryland, College Park
*http://www.norbertwiener.umd.edu*

# Narrow band ambiguity functions and CAZAC codes

# Discrete ambiguity functions

Let $u : \{0, 1, \ldots, N-1\} \to \mathbb{C}$.

- $u_p : \mathbb{Z}_N \to \mathbb{C}$ is the $N$-periodic extension of $u$.
- $u_a : \mathbb{Z} \to \mathbb{C}$ is an aperiodic extension of $u$:

$$u_a[m] = \begin{cases} u[m], & m = 0, 1, \ldots, N-1 \\ 0, & \text{otherwise.} \end{cases}$$

- The *discrete periodic ambiguity function* $A_p(u) : \mathbb{Z}_N \times \mathbb{Z}_N \to \mathbb{C}$ of $u$ is

$$A_p(u)(m, n) = \frac{1}{N} \sum_{k=0}^{N-1} u_p[m+k] \overline{u_p[k]} e^{2\pi ikn/N}.$$

- The *discrete aperiodic ambiguity* function $A_a(u) : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ of $u$ is

$$A_a(u)(m, n) = \frac{1}{N} \sum_{k=0}^{N-1} u_a[m+k] \overline{u_a[k]} e^{2\pi ikn/N}.$$

# CAZAC sequences

- $u : \mathbb{Z}_N \to \mathbb{C}$ is *Constant Amplitude Zero Autocorrelation (CAZAC)*:

$$\forall m \in \mathbb{Z}_N, \quad |u[m]| = 1, \text{ (CA)}$$

  and

$$\forall m \in \mathbb{Z}_N \setminus \{0\}, \quad A_p(u)(m,0) = 0. \text{ (ZAC)}$$

- Empirically, the (ZAC) property of CAZAC sequences $u$ leads to phase coded waveforms $w$ with low *aperiodic autocorrelation* $\mathcal{A}(w)(t,0)$.
- Are there only finitely many non-equivalent CAZAC sequences?
  - "Yes" for $N$ prime and "No" for $N = MK^2$,
  - Generally unknown for $N$ square free and not prime.

# Björck CAZAC codes and ambiguity function comparisons

# Legendre symbol

Let $N$ be a prime and $(k, N) = 1$ .

- $k$ is a quadratic residue mod $N$ if $x^2 = k \pmod N$ has a solution.
- $k$ is a quadratic non–residue mod $N$ if $x^2 = k \pmod N$ has no solution.
- The Legendre symbol:

$$\left(\frac{k}{N}\right) = \begin{cases} 1, & \text{if } k \text{ is a quadratic residue mod } N\ , \\ -1, & \text{if } k \text{ is a quadratic non–residue mod } N. \end{cases}$$

The diagonal of the product table of $\mathbb{Z}_N$ gives values $k \in \mathbb{Z}$ which are squares. As such we can program Legendre symbol computation.

Example: $N = 7$. $(\frac{k}{N}) = 1$ if $k = 1, 2, 4$.

# Definition

Let $N$ be a prime number. A *Björck CAZAC sequence* of length $N$ is

$$u[k] = e^{i\theta_N(k)}, \quad k = 0, 1, \ldots, N-1,$$

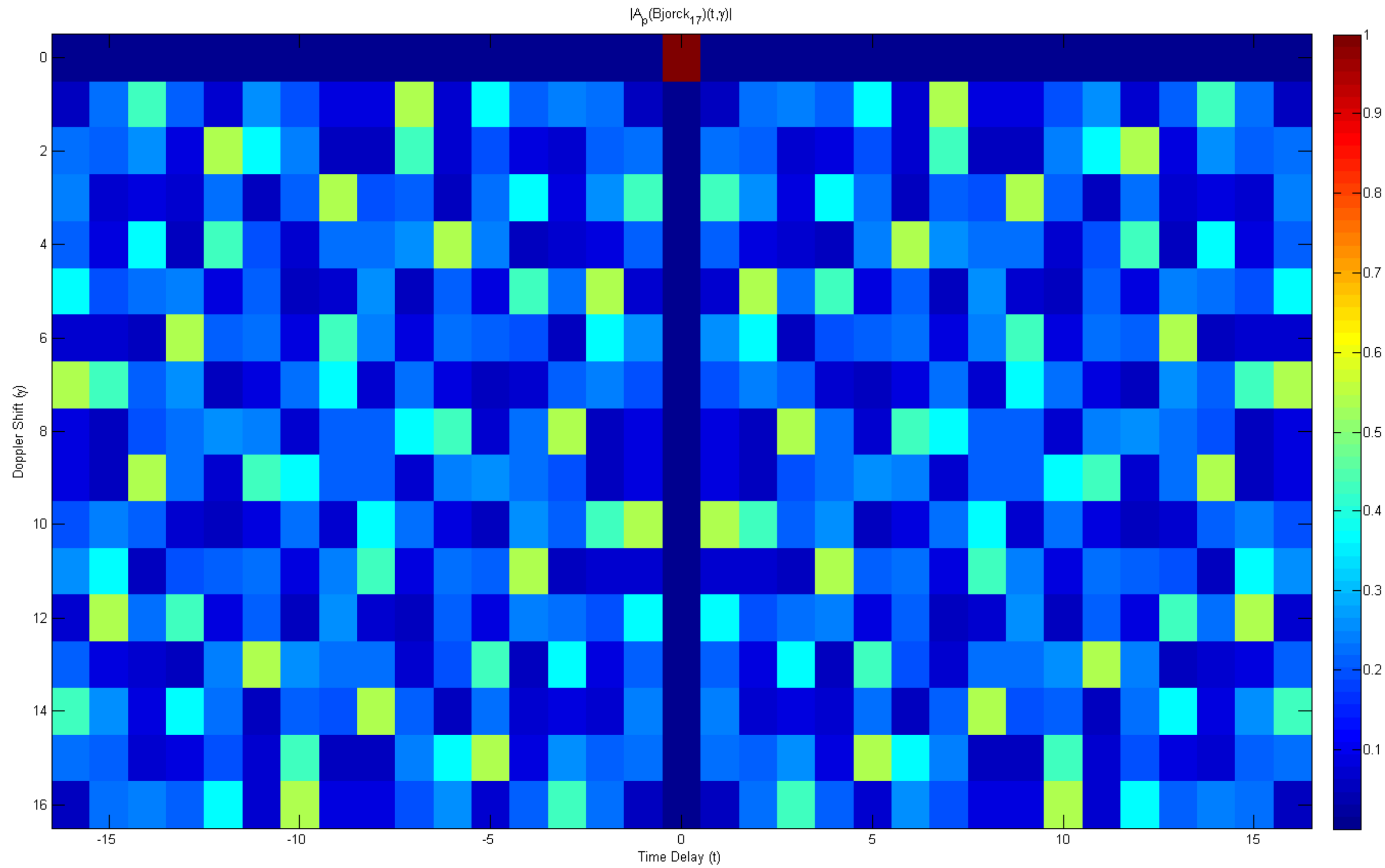
where, for $N = 1 \pmod 4$,

$$\theta_N(k) = \arccos\left(\frac{1}{1+\sqrt{N}}\right)\left(\frac{k}{N}\right),$$

and, for $N = 3 \pmod 4$,

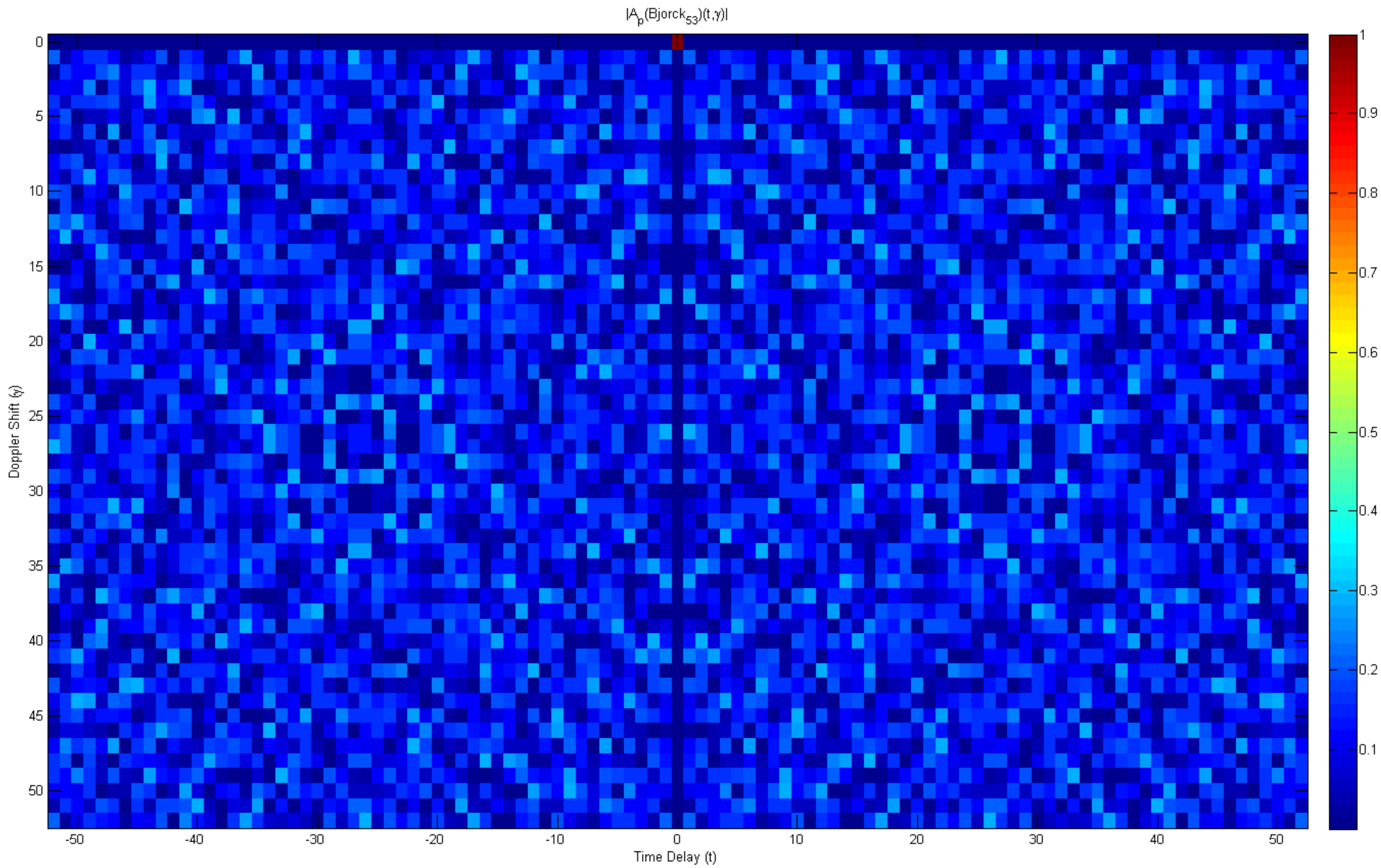
$$\theta_N(k) = \frac{1}{2}\arccos\left(\frac{1-N}{1+N}\right)\left[(1-\delta_k)\left(\frac{k}{N}\right) + \delta_k\right].$$

$\delta_k$ is Kronecker delta and $\left(\frac{k}{N}\right)$ is Legendre symbol.

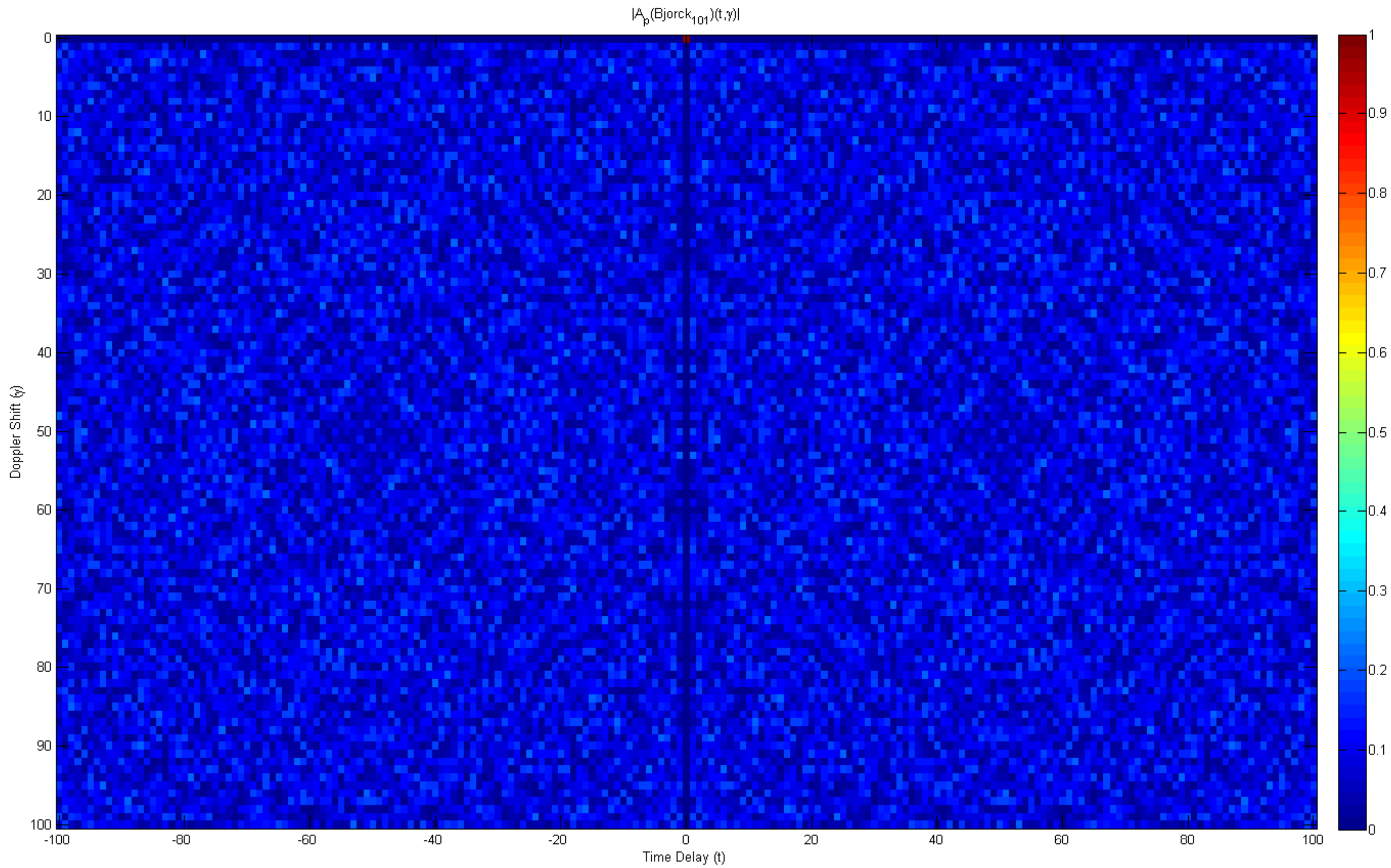



Absolute value of Bjorck code of length 17

Norbert Wiener Center
for Harmonic Analysis and Applications

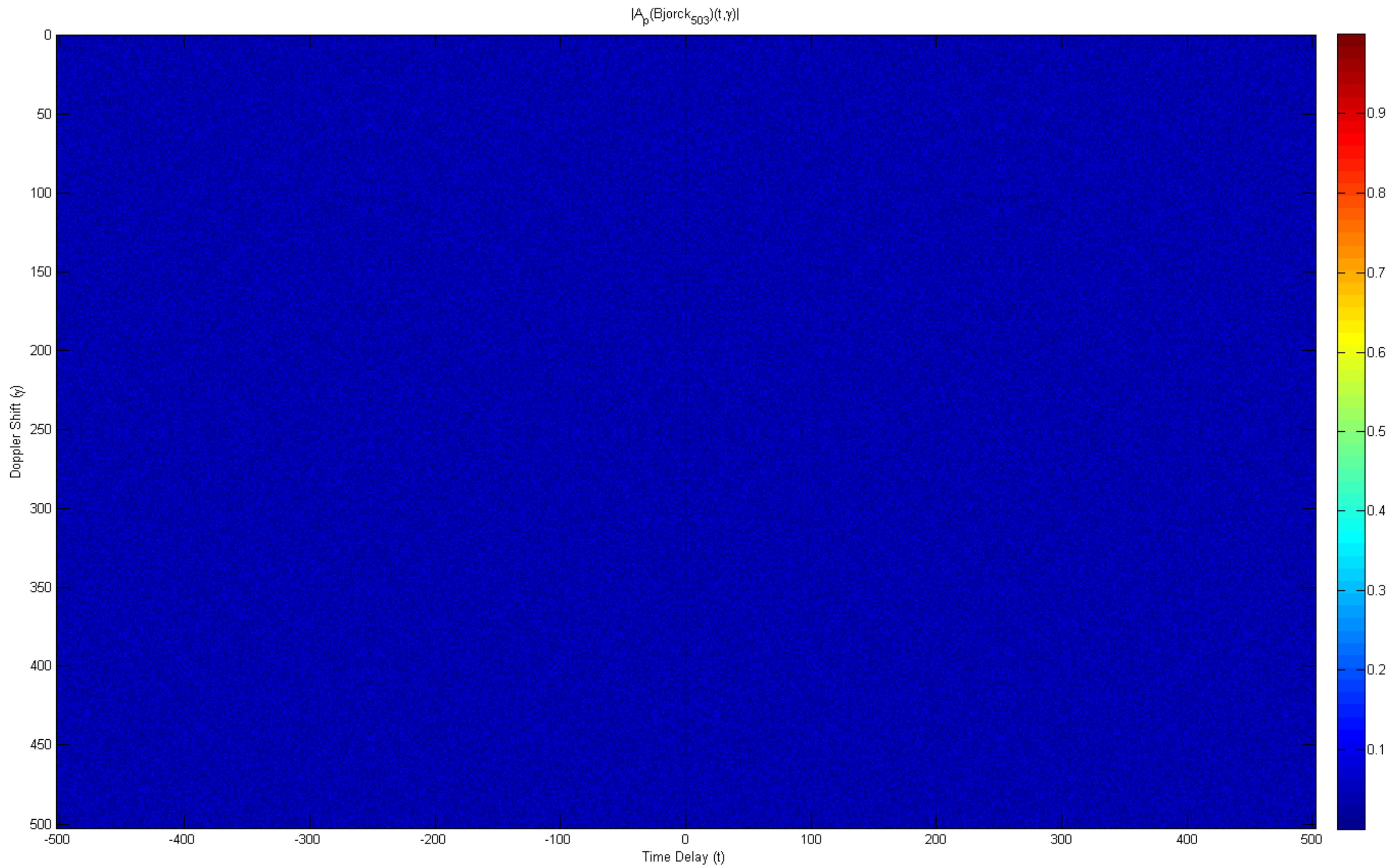



Absolute value of Bjorck code of length 53



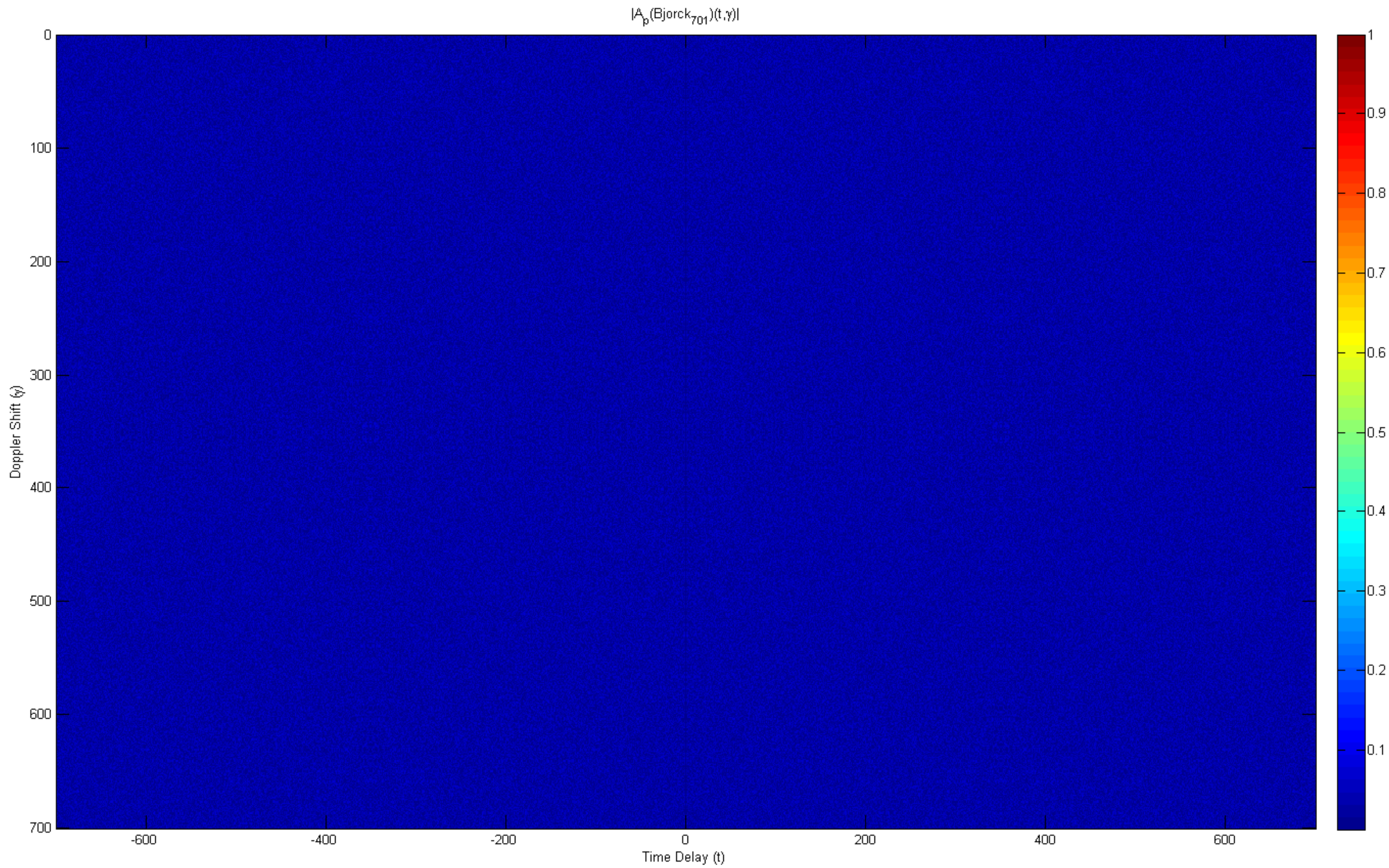

Absolute value of Bjorck code of length 101

Norbert Wiener Center for Harmonic Analysis and Applications



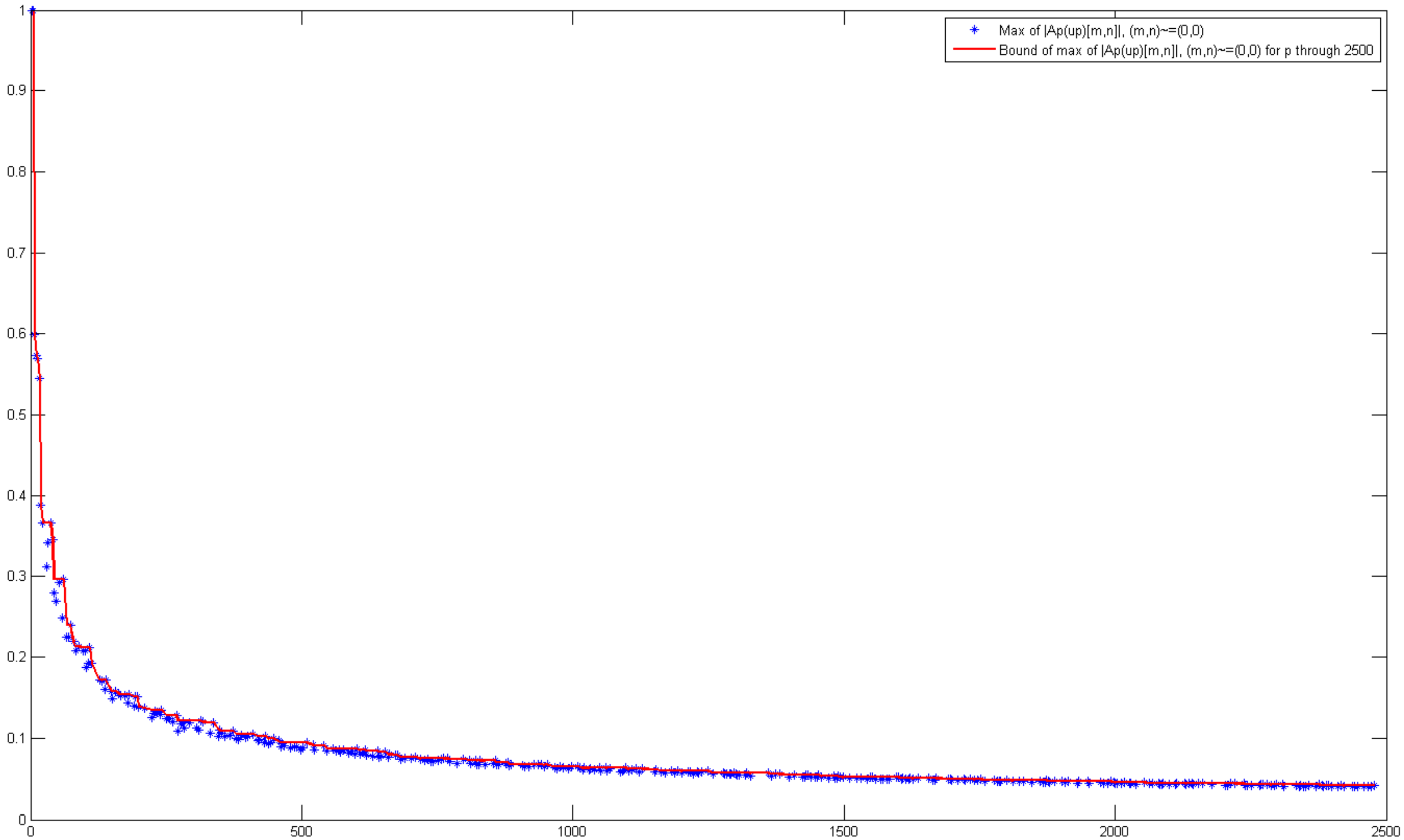

Absolute value of Bjorck code of length 503

Norbert Wiener Center for Harmonic Analysis and Applications




Absolute value of Bjorck code of length 701

Norbert Wiener Center for Harmonic Analysis and Applications




Norbert Wiener Center
for Harmonic Analysis and Applications

# Björck CAZAC Discrete Narrow-band Ambiguity Function

Let $u_p$ denote the Björck CAZAC sequence for prime $p$, and let $A_p(u_p)$ be the discrete narrow band ambiguity function defined on $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

**Theorem (J. and R. Benedetto and J. Woodworth)**

$$|A_p(u_p)(m,n)| \leq \frac{2}{\sqrt{p}} + \frac{4}{p}$$

for all $(m,n) \in (\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}) \setminus (0,0)$.

- The bound is more precise but not better than $\frac{2}{\sqrt{p}}$ depending on whether $p \equiv 1 \pmod 4$ or $p \equiv 3 \pmod 4$.
- The proof is at the level of Weil's proof of the Riemann hypothesis for finite fields and depends on Weil's exponential sum bound.
- Elementary construction/coding and intricate combinatorial/geometrical patterns.

Waveform diversity is a government program for disadvantaged waveforms

– G. Linde, (a real) radar engineer

# The ambiguity function

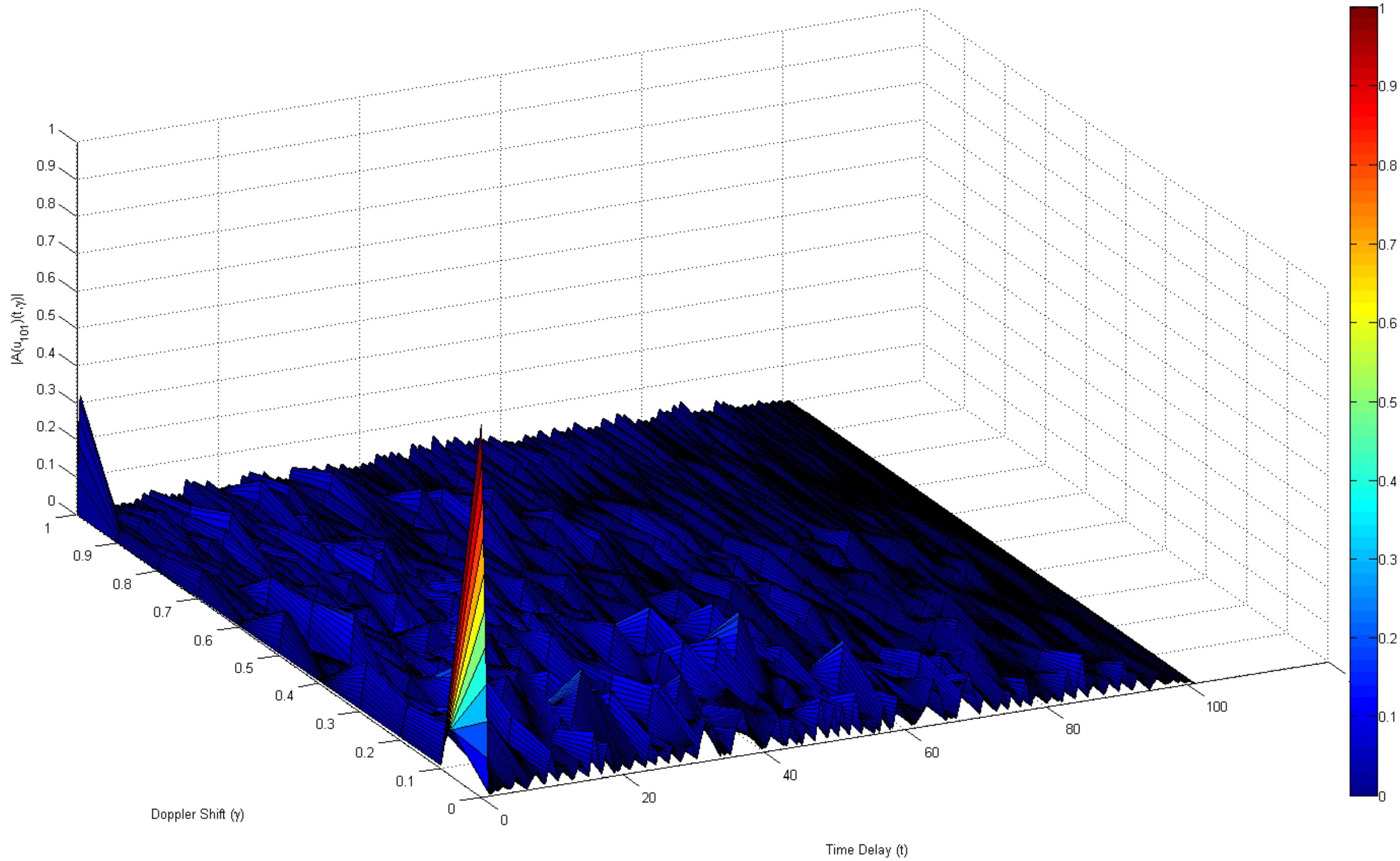
- The *complex envelope* $w$ of the *phase coded waveform* $Re(w)$ associated to a unimodular N-periodic sequence $u : \mathbb{Z}_N \to \mathbb{C}$ is

$$w(t) = \frac{1}{\sqrt{\tau}} \sum_{k=0}^{N-1} u[k]\, \mathbb{1}\left(\frac{t - kt_b}{t_b}\right),$$

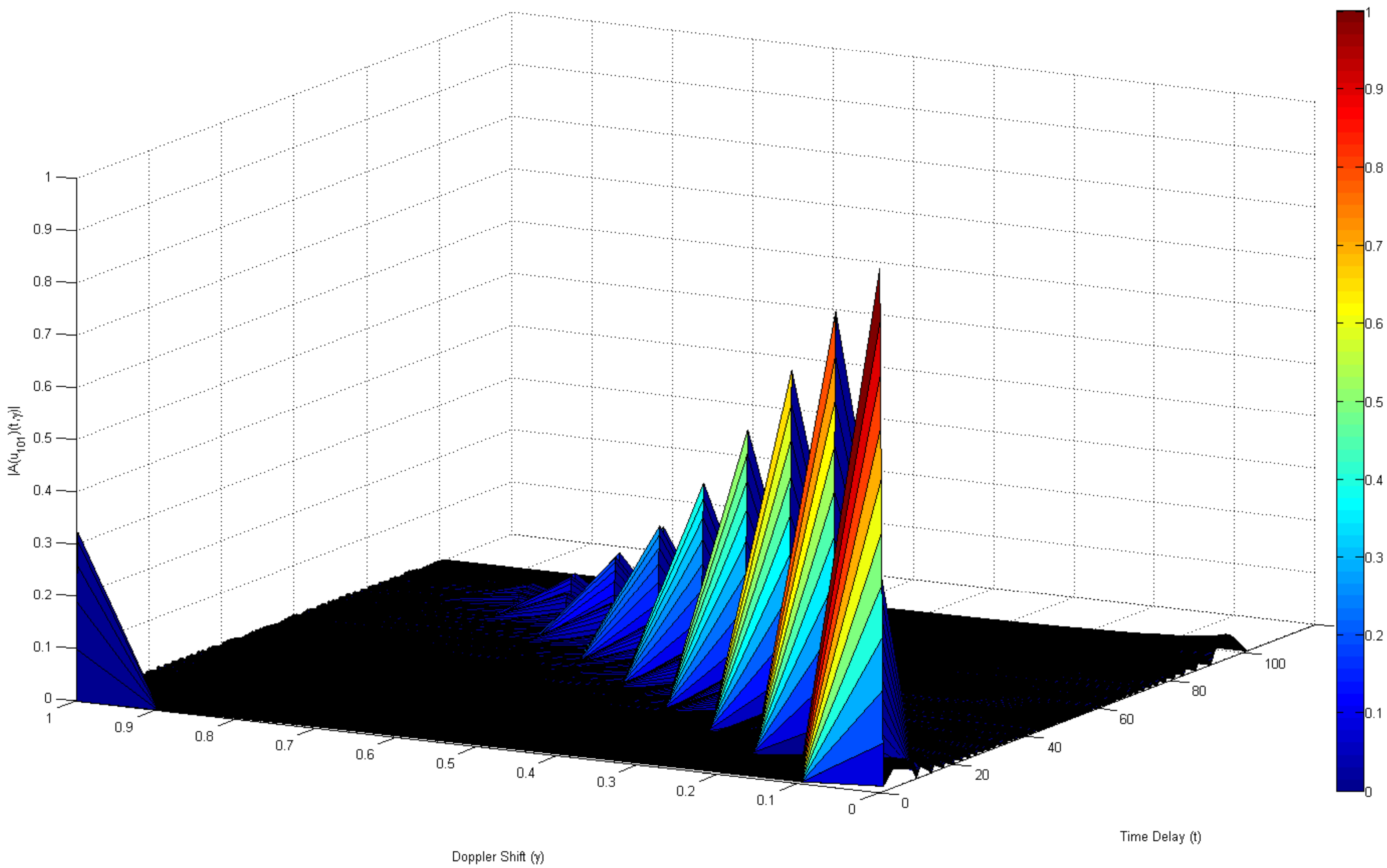
  where $\mathbb{1}$ is the characteristic function of the interval $[0, 1)$, $\tau$ is the pulse duration, and $t_b = \tau/N$.
- For spectral shaping problems, smooth replacements to $\mathbb{1}$ are analyzed.
- The *(aperiodic) ambiguity function* $\mathcal{A}(w)$ of $w$ is

$$\mathcal{A}(w)(t,\gamma) = \int w(s+t)\overline{w(s)}e^{2\pi i s\gamma}\, ds,$$

  where $t \in \mathbb{R}$ is time delay and $\gamma \in \widehat{\mathbb{R}}(= \mathbb{R})$ is frequency shift.




Norbert Wiener Center
for Harmonic Analysis and Applications


|A(u_101)(t,γ)|
Doppler Shift (γ)
Time Delay (t)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
20
40
60
80
100

# Perspective

Sequences for coding theory, cryptography, phase-coded waveforms, and communications (synchronization, fast start-up equalization, frequency hopping) include the following in the periodic case:

- Gauss, Wiener (1927), Zadoff (1963), Schroeder (1969), Chu (1972), Zhang and Golomb (1993)
- Frank (1953), Zadoff and Abourezk (1961), Heimiller (1961)
- Milewski (1983)
- Björck (1985) and Golomb (1992),

and their generalizations, both periodic and aperiodic.

The general problem of using codes to generate signals leads to frames.

Norbert Wiener Center
for Harmonic Analysis and Applications

# Balayage, Fourier frames, and sampling theory

Norbert Wiener Center
for Harmonic Analysis and Applications

# Fourier frames, goal, and a litany of names

**Definition**

E $= \{x_n\} \subseteq \mathbb{R}^d, \Lambda \subseteq \widehat{\mathbb{R}}^d$. E is a *Fourier frame* for $L^2(\Lambda)$ if

$\exists A, B > 0, \forall F \in L^2(\Lambda)$,

$$A \; ||F||^2_{L^2(\Lambda)} \leq \sum_n | < F(\gamma), e^{-2\pi i x_n \cdot \gamma} > |^2 \leq B \; ||F||^2_{L^2(\Lambda)}.$$

- *Goal* Formulate a general theory of Fourier frames and non-uniform sampling formulas parametrized by the space $M(\mathbb{R}^d)$ of bounded Radon measures.
- *Motivation* Beurling theory (1959-1960).
- *Names* Riemann-Weber, Dini, G.D. Birkhoff, Paley-Wiener, Levinson, Duffin-Schaeffer, Beurling-Malliavin, Beurling, H.J. Landau, Jaffard, Seip, Ortega-Certà–Seip.

- Let $M(G)$ be the algebra of bounded Radon measures on the LCAG $G$.
- Balayage in potential theory was introduced by Christoffel (early 1870s) and Poincaré (1890).

**Definition**

(Beurling) Balayage is possible for $(E, \Lambda) \subseteq G \times \widehat{G}$, a LCAG pair, if

$$\forall \mu \in M(G), \exists \nu \in M(E) \text{ such that } \hat{\mu} = \hat{\nu} \text{ on } \Lambda.$$

We write balayage $(E, \Lambda)$.

- The set, $\Lambda$, of group characters is the analogue of the original role of $\Lambda$ in balayage as a collection of potential theoretic kernels.
- Kahane formulated balayage for the harmonic analysis of restriction algebras.

Norbert Wiener Center for Harmonic Analysis and Applications

# Spectral synthesis

**Definition**

(Wiener, Beurling) Closed $\Lambda \subseteq \widehat{G}$ is a set of *spectral synthesis* (S-set) if

$$\forall \mu \in M(G), \forall f \in C_b(G),$$
$$\text{supp}(\widehat{f}) \subseteq \Lambda \text{ and } \hat{\mu} = 0 \text{ on } \Lambda \implies \int_G f \, d\mu = 0.$$

$$(\forall T \in A'(\widehat{G}), \forall \phi \in A(\widehat{G}), \quad \text{supp}(T) \subseteq \Lambda \text{ and } \phi = 0 \text{ on } \Lambda \Rightarrow T(\phi) = 0.)$$

- Ideal structure of $L^1(G)$ - the Nullstellensatz of harmonic analysis
- $T \in D'(\widehat{\mathbb{R}}^d), \phi \in C_c^\infty(\widehat{\mathbb{R}}^d)$, and $\phi = 0$ on $\text{supp}(T) \Rightarrow T(\phi) = 0$, with same result for $M(\widehat{\mathbb{R}}^d)$ and $C_0(\widehat{\mathbb{R}}^d)$.
- $S^2 \subseteq \widehat{\mathbb{R}}^3$ is not an S-set (L. Schwartz), and every non-discrete $\widehat{G}$ has non-S-sets (Malliavin).
- Polyhedra are S-sets. The $\frac{1}{3}$-Cantor set is an S-set with non-S-subsets.

# Strict multiplicity

## Definition

$\Gamma \subseteq \widehat{G}$ is a set of *strict multiplicity* if

$$\exists\ \mu \in M(\Gamma)\backslash\{0\} \text{ such that } \check{\mu} \text{ vanishes at infinity in } G.$$

- Riemann and sets of uniqueness in the wide sense.
- Menchov (1916): $\exists$ closed $\Gamma \subseteq \widehat{\mathbb{R}}/\mathbb{Z}$ and $\mu \in M(\Gamma)\backslash\{0\}$,
$$|\Gamma| = 0 \text{ and } \check{\mu}(n) = O((\log|n|)^{-1/2}), |n| \to \infty.$$
- 20th century history to study rate of decrease: Bary (1927), Littlewood (1936), Salem (1942, 1950), Ivašev-Mucatov (1957), Beurling.

## Assumption

*$\forall\ \gamma \in \Lambda$ and $\forall\ N(\gamma)$, compact neighborhood, $\Lambda \cap N(\gamma)$ is a set of strict multiplicity.*

# A theorem of Beurling

**Definition**

E $= \{x_n\} \subseteq \mathbb{R}^d$ is *separated* if

$$\exists\, r > 0,\ \forall m, n,\ m \neq n \Rightarrow ||x_m - x_n|| \geq r.$$

**Theorem**

*Let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be a compact S-set, symmetric about $0 \in \widehat{\mathbb{R}}^d$, and let E $\subseteq \mathbb{R}^d$ be separated. If balayage (E,$\Lambda$), then*

*E is a Fourier frame for $L^2(\Lambda)$.*

- Equivalent formulation in terms of

$$PW_\Lambda = \{f \in L^2(\mathbb{R}^d) : \mathrm{supp}(\hat{f}) \subseteq \Lambda\}.$$

- $\forall F \in L^2(\Lambda), \qquad F = \sum_{x \in \mathrm{E}} < F, S^{-1}(e_x) >_\Lambda e_x$ in $L^2(\Lambda)$.
- For $\mathbb{R}^d$ and other generality beyond Beurling's theorem in $\mathbb{R}$, the result above was formulated by Hui-Chuan Wu and JB (1998), see Landau (1967).

Let $G \in L^2(\widehat{\mathbb{R}}^d)$ satisfy $||G||_{L^2(\widehat{\mathbb{R}}^d)} = 1$; let $\Lambda \subset \widehat{\mathbb{R}}^d$ be an S-set, symmetric about 0; and let $E \subset \mathbb{R}^d$ be separated. Define

(STFT) $\quad \forall F \in L^2(\Lambda), \quad V_G F(x,\gamma) = \int_\Lambda F(\lambda)G(\lambda-\gamma)e^{2\pi i x\cdot\lambda}\, d\lambda.$

**Theorem**

*If balayage (E,$\Lambda$), then*

$$\exists A, B > 0, \quad \forall F \in L^2(\Lambda),$$

$$A\ ||F||^2_{L^2(\Lambda)} \leq \int_{\widehat{\mathbb{R}}^d} \sum_{x\in E} |V_G F(x,\gamma)|^2\, d\gamma \leq B\ ||F||^2_{L^2(\Lambda)}.$$

*Remark* There are basic problems to be resolved and there have been fundamental recent advances.

Norbert Wiener Center
for Harmonic Analysis and Applications

# Examples of balayage

1. Let E $\subseteq \mathbb{R}^d$ be separated. Define

$$r = r(E) = \sup_{x \in \mathbb{R}^d} \text{dist}(x, E).$$

   If $r\rho < \frac{1}{4}$, then balayage (E, $\bar{B}(0, \rho)$). $\frac{1}{4}$ is the best possible.

2. If balayage (E, $\Lambda$) and $\Lambda_0 \subseteq \Lambda$, then balayage (E, $\Lambda_0$).
3. Let E $= \{x_n\}$ be a Fourier frame for PW$_\Lambda$. Then for all $\Lambda_0 \subseteq \Lambda$ with dist$(\Lambda_0, \Lambda^c\ ) > 0$, we have balayage (E, $\Lambda_0$).
4. In $\mathbb{R}^1$, for a separated set E, Beurling lower density $> \rho$ is necessary and sufficient for balayage (E, $[\frac{-\rho}{2}, \frac{\rho}{2}]$).

*Remark* In $\mathbb{R}^1$, if E is uniformly dense in the sense of Duffin-Schaeffer, then $D^-(E), D^+(E)$, and $D_u(E)$ coincide.
So Beurling's result $\Rightarrow$ Duffin-Schaeffer's result on Fourier frames.

Norbert Wiener Center for Harmonic Analysis and Applications

*ΦDOs and the Kohn-Nirenberg correspondence*

**Definition/notation for $\Lambda \subseteq \widehat{\mathbb{R}}^d$**

- $\forall \gamma \in \Lambda$, $g_\gamma \in C_b(\mathbb{R}^d)$ and $\widehat{supp(g_\gamma)} \subseteq \Lambda$
- $s(x,\gamma) = e^{2\pi i x\cdot\gamma} g_\gamma(x)$

The *Kohn-Nirenberg correspondence*

$$s \longmapsto H_s$$

with *symbol* $H_s$ is defined by the *Hörmander operator*

$$H_s : L^2(\widehat{\mathbb{R}}^d) \to L^2(\Lambda) \subseteq L^2(\widehat{\mathbb{R}}^d)$$

$$H_s(\hat{f})(\gamma) = \int_{\mathbb{R}^d} s(x,\gamma) f(x) e^{-2\pi i x\cdot\gamma}\, dx$$

**Remark**

Classically, the symbol is $\sigma$ and integration is over $\widehat{\mathbb{R}}^d$.

*ΦDOs and generalized Fourier frames for non-uniform sampling*

**Theorem**

Assume balayage $(E, \Lambda)$ where $\Lambda \subseteq \widehat{\mathbb{R}}^d$ is a compact, symmetric S-set. Assume $E = \{x_n\}$ is separated. Let $s(x,\gamma) = e^{2\pi i x\cdot\gamma} g_\gamma(x)$, where

$$\{g_\gamma : \gamma \in \Lambda\} \subseteq C_b(\mathbb{R}^d)$$

and

$$\forall \gamma \in \Lambda, \quad supp(\widehat{g_\gamma}) \subseteq \Lambda$$

Let $f \in X_s \subseteq L^2(\mathbb{R}^d)$ if $H_s(\widehat{f}) = F \in L^2(\Lambda)$ and $supp F \subseteq \Lambda$, then

$$\exists A > 0 \quad \text{such that} \quad \forall f \in X_s$$

$$A \frac{\int_\Lambda |F(\gamma)|^2 \, d\gamma}{\|f\|_{L^2(\mathbb{R}^d)}} \leq \left( \sum_{n\in\mathbb{Z}} \left| \int_\Lambda \overline{F(\gamma)} s(x_n,\gamma) e^{2\pi i x_n\cdot\gamma} \, d\gamma \right|^2 \right)^{1/2}.$$

Classification

- Dimension reduction
- Finite frames and frame potential energy
- Frame potential energy classification algorithm
- Hyperspectral image processing

# Dimension reduction

# Kernel dimension reduction

Given data space $X$ of $N$ vectors in $\mathbb{R}^D$. ($N$ is the number of pixels in the hypercube, $D$ is the number of spectral bands.)

Two Steps:

1. Construction of an $N \times N$ symmetric, positive semi–definite kernel, $K$, from these $N$ data points in $\mathbb{R}^D$.
2. Diagonalization of $K$, and then choosing $d \leq D$ *significant orthogonal* eigenmaps of $K$.

- Different classes of interest may not be orthogonal to each other; however, they may be captured by different frame elements. It is plausible that classes may correspond to elements in a frame but not elements in a basis.
- A *frame* generalizes the concept of an orthonormal basis. Frame elements are non–orthogonal.

Norbert Wiener Center for Harmonic Analysis and Applications

# Dimension reduction paradigm

- Given data space $X$ of $N$ vectors $x_m \in \mathbb{R}^D$, and let

$$K : X \times X \to \mathbb{R}$$

  be a symmetric ($K(x, y) = K(y, x)$), positive semi–definite kernel.
- We map $X$ to a low dimensional space via the following mapping:

$$X \longrightarrow K \longrightarrow \mathbb{R}^d(K), \quad d < D$$

$$x_m \mapsto y_m = (y[m, n_1], y[m, n_2], \ldots, y[m, n_d]) \in \mathbb{R}^d(K),$$

  where $y[\cdot, n] \in \mathbb{R}^N$ is an eigenvector of $K$.

Norbert Wiener Center for Harmonic Analysis and Applications

# Laplacian Eigenmaps

- Consider the data points $X$ as the nodes of a graph.
- Define a metric $\rho : X \times X \longrightarrow \mathbb{R}^+$, e.g., $\rho(x_m, x_n) = \|x_m - x_n\|$ is the Euclidean distance.
- Choose $q \in \mathbb{N}$.
- For each $x_i$ choose the $q$ nodes $x_n$ closest to $x_i$ in the metric $\rho$, and place an edge between $x_i$ and each of these nodes.
- This defines $N'(x_i)$, viz.,
  $N'(x_i) = \{x \in X \ : \ \exists \text{ an edge between } x \text{ and } x_i.\}$.
- To define the weights on the edges, we compute:

$$W_{ij} = \begin{cases} \exp(-\|x_i - x_j\|^2/\sigma) & \text{if } x_j \in N'(x_i) \text{ or } x_i \in N'(x_j) \\ 0 & \text{otherwise} \end{cases}$$

- Set $K = D - W$, where $D_{ii} = \sum_j W_{ij}$ and $D_{ij} = 0$ for $i \neq j$;
- Diagonalize $K$.
- $K$ is symmetric and positive semi–definite.

Norbert Wiener Center for Harmonic Analysis and Applications

# Finite frames and frame potential energy

# FUNTF

- A set $F = \{e_j\}_{j\in J} \subseteq \mathbb{F}^d$ is a *frame* for $\mathbb{F}^d$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, if

$$\exists\, A, B > 0 \quad \text{such that} \quad \forall\, x \in \mathbb{F}^d, \quad A\|x\|^2 \leq \sum_{j\in J} |\langle x, e_j\rangle|^2 \leq B\|x\|^2.$$

- $F$ *tight* if $A = B$. A finite unit-norm tight frame $F$ is a FUNTF.
- $N$ row vectors from any fixed $N \times d$ submatrix of the $N \times N$ DFT matrix, $\frac{1}{\sqrt{d}}(e^{2\pi imn/N})$, is a FUNTF for $\mathbb{C}^d$.
- If $F$ is a FUNTF for $\mathbb{F}^d$, then

$$\forall x \in \mathbb{F}^d, \quad x = \frac{d}{N}\sum_{j=1}^{N} \langle x, e_j\rangle e_j.$$

- Frames: redundant representation, compensate for hardware errors, inexpensive, numerical stability, minimize effects of noise.

# CAZACs and FUNTFs

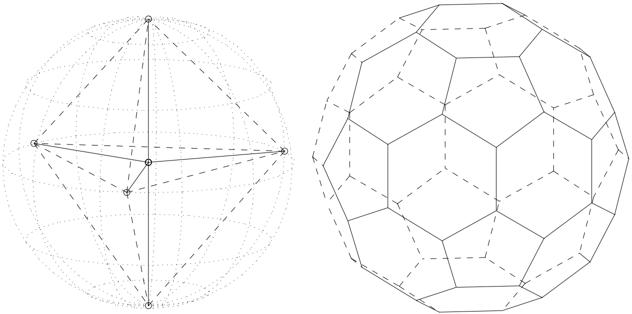
- Let $u = \{u[k]\}_{k=1}^{N}$ be a CAZAC sequence in $\mathbb{C}$. Define

$$\forall\, k = 1, ..., N, \quad v_k = v[k] = \frac{1}{\sqrt{d}}(u[k], u[k+1], ..., u[k+d-1]).$$

  Then $v = \{v[k]\}_{k=1}^{N} \subseteq \mathbb{C}^d$ is a CAZAC sequence in $\mathbb{C}^d$ and $\{v_k\}_{k=1}^{N}$ is a FUNTF for $\mathbb{C}^d$ with frame constant $N/d$.

- Let $\{x_k\}_{k=1}^{N} \subseteq \mathbb{C}^d$ be a FUNTF for $\mathbb{C}^d$, with frame constant $A$ and with associated Bessel map $L : \mathbb{C}^d \to \ell^2(\mathbb{Z}_N)$; and let $u = \{u[j]\}_{j=1}^{M} \subseteq \mathbb{C}^d$ be a CAZAC sequence in $\mathbb{C}^d$. Then $\{\frac{1}{\sqrt{A}} L(u[j])\}_{j=1}^{M} \subseteq \mathbb{C}^N (= \ell^2(\mathbb{Z}_N))$ is a CAZAC sequence in $\mathbb{C}^N$.

# Examples of frames



(a) Non–FUNTF (b) FUNTF

# Frame force and potential energy

$$F : S^{d-1} \times S^{d-1} \setminus D \longrightarrow \mathbb{R}^d$$

$$P : S^{d-1} \times S^{d-1} \setminus D \longrightarrow \mathbb{R},$$

where $P(a,b) = p(\|a-b\|), \qquad p'(x) = -xf(x)$

- Coulomb force

$$CF(a,b) = (a-b)/\|a-b\|^3, \quad f(x) = 1/x^3$$

- Frame force

$$FF(a,b) = \langle a,b\rangle(a-b), \quad f(x) = 1 - x^2/2$$

- Total potential energy for the frame force

$$TFP(\{x_n\}) = \sum_{m=1}^{N}\sum_{n=1}^{N} |\langle x_m, x_n\rangle|^2$$

# Characterization of FUNTFs

**Theorem**

Let $N \leq d$. The minimum value of *TFP*, for the frame force and $N$ variables, is $N$; and the *minimizers* are precisely the orthonormal sets of $N$ elements for $\mathbb{R}^d$.

Let $N \geq d$. The minimum value of *TFP*, for the frame force and $N$ variables, is $N^2/d$; and the *minimizers* are precisely the FUNTFs of $N$ elements for $\mathbb{R}^d$.

**Problem**

Find FUNTFs analytically, effectively, computationally.

Frame potential energy classification algorithm

# Optimization problem: maximal separation

Goal: Construct a FUNTF $\{\Psi_k\}_{k=1}^s$ such that each $\Psi_k$ is associated to only one classifiable material.

For $\{\theta_k\}_{k=1}^s \in S^{d-1} \times \cdots \times S^{d-1}$ and $n = 1, \ldots, s$, set

$$p(\theta_n) = \sum_{m=1}^{N} |\langle y_m, \theta_n \rangle|$$

and consider the maximal separation

$$\sup_{\{\theta_j\}_{j=1}^s} \min\{|p(\theta_k) - p(\theta_n)| : k \neq n\}.$$

# Optimization problem: ideal class definition

- $Y = \{y_m\}_{m=1}^N \subseteq \mathbb{R}^d(K) \subseteq \mathbb{R}^N$
- Given $s$ classes $C_j, j = 1, \ldots, s$, defined in terms of a tolerance $\epsilon > 0$ and partition $\{P_j\}_{j=0}^s$ of $Y$, as $C_j = P_j \cap Y \subseteq B(z_j, \epsilon)$ $j = 1, \ldots, s$ for some $z_j \in \mathbb{R}^d(K)$.

# Optimization problem: FUNTF construction

- Point of view: Combine frame potential energy theorem, maximal separation criteria ($M_\delta$), and ideal class definition ($C_\epsilon$).

- Paradigm: Given $Y$, $s$, $M_\delta$, and $C_\epsilon$. Construct a FUNTF $\{\Psi_j\}_{j=1}^{s}$ such that

$$\forall j = 1, \ldots, s, \quad |\langle \Psi_j, Y \cap P_j \rangle| \geq R(\epsilon, \delta)$$

and

$$\forall j \neq k, \quad |\langle \Psi_k, Y \cap P_j \rangle| \leq r(\epsilon, \delta),$$

where $r(\epsilon, \delta) < R(\epsilon, \delta)$.

# Frame coefficient images

- Given $\Psi = \{\Psi_n\}_{n=1}^{s} \subseteq \mathbb{R}^d = \mathbb{R}^d(K) \subseteq \mathbb{R}^N$ and $m \in \{1, \ldots, N\}$. Consider the set of frame decompositions

$$\forall y_m \in \mathbb{R}^d, m = 1, \ldots, N, \quad y_m = \sum_{n=1}^{s} c_{m,n}^{\alpha} \Psi_n, \quad \text{indexed by } \alpha \in \mathbb{R}.$$

- For each $m \in \{1, \ldots, N\}$ choose an $\ell^1$ sparse decomposition

$$y_m = \sum_{n=1}^{s} c_{m,n}^{\alpha(m)} \Psi_n$$

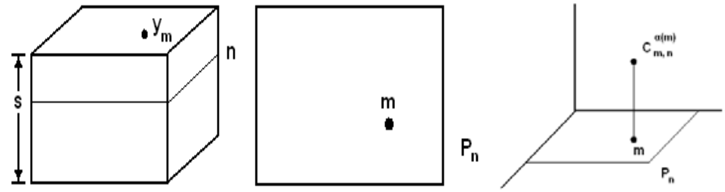
defined by the inequality,

$$\forall \alpha, \quad \sum_{n=1}^{s} |c_{m,n}^{\alpha(m)}| \leq \sum_{n=1}^{s} |c_{m,n}^{\alpha}|.$$

Norbert Wiener Center
for Harmonic Analysis and Applications

# Frame coefficient images (continued)

- Choose $n \in \{1, ..., s\}$. Take a slice, $P_n$, of the data cube at $n$. $P_n$ contains $N$ points $m$.




(a) Data Cube (b) Top Down Slice (c) $c_{m,n}^{\alpha(m)}$ defined

- The image with $N$ pixels $m$, associated to the the frame element $\Psi_n$, is defined by $\{c_{m,n}^{\alpha(m)} \mid m = 1, ..., N\}$.

Norbert Wiener Center for Harmonic Analysis and Applications

# Hyperspectral image processing

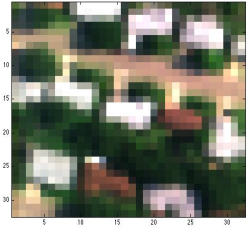
# Urban data set classes


Figure: HYDICE Copperas Cove, TX — http://www.tec.army.mil/Hypercube/

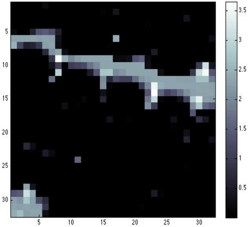
# Urban data set classes

- There are 23 classes associated with the different colors in the previous figure.
- In fact, if the 23 classes were to correspond roughly to orthogonal subspaces, then one cannot achieve effective dimension reduction less than dimension $d = 23$.
- However, we could have a frame with 23 elements in a space of reduced dimension $d < 23$.

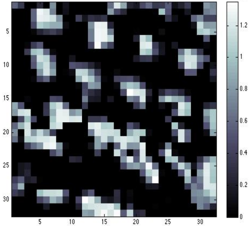
# Frame coefficients



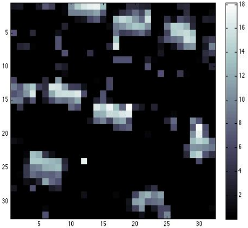

(a) Original




(b) Road coefficients



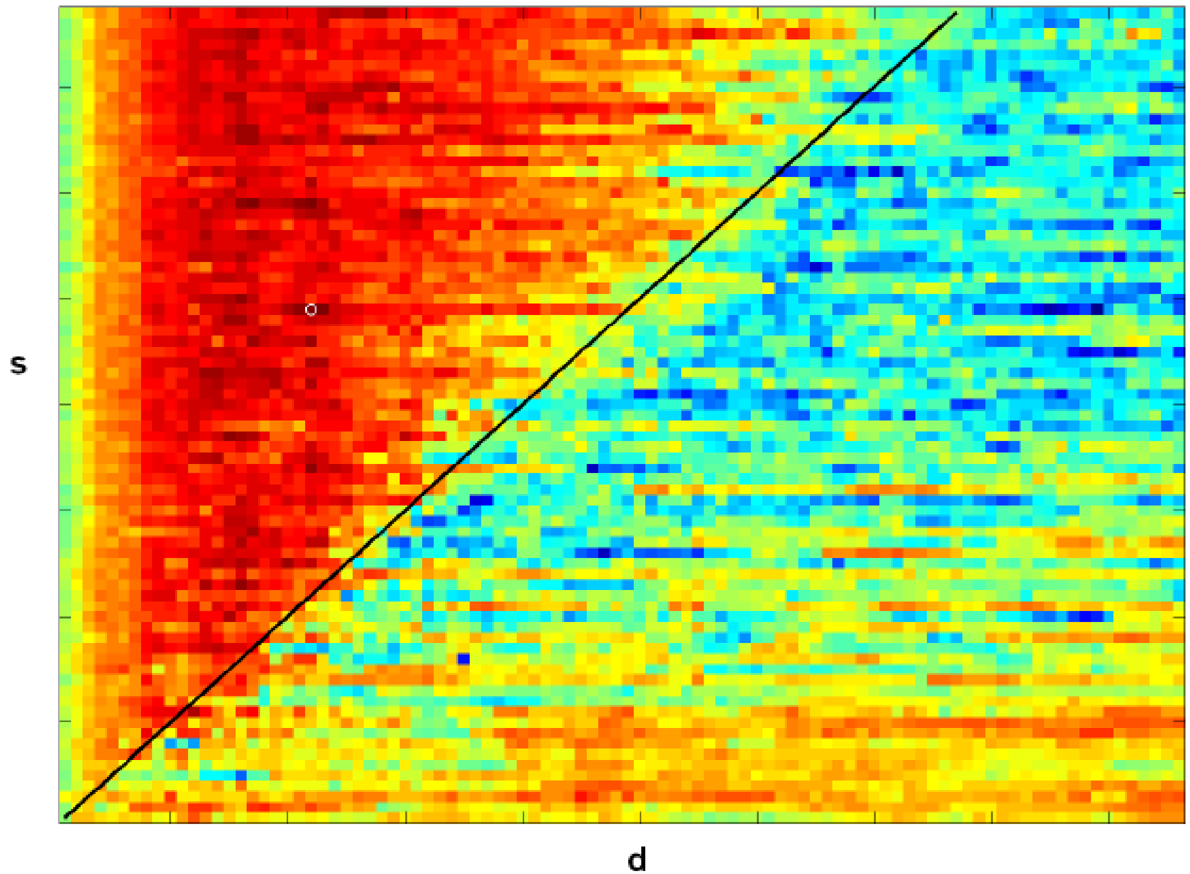

(c) Tree coefficients




(d) White house coefficients

Norbert Wiener Center
for Harmonic Analysis and Applications

# Overview of Classification Results



# Happy 60th, Hans!

That's all folks!

Norbert Wiener Center
for Harmonic Analysis and Applications

Norbert Wiener Center
for Harmonic Analysis and Applications

# Sampling formulas (1)

- Let $\Lambda \in \widehat{\mathbb{R}}^d$ be a compact S-set, and assume balayage (E, $\Lambda$), E $= \{x_n\} \subseteq \mathbb{R}^d$ separated.
- *Theorem* $\exists \epsilon > 0$, balayage (E, $\Lambda_\epsilon$).
- *Theorem* $\forall x \in \mathbb{R}^d,\ \exists\ \{b_n(x)\} \in l^1(\mathbb{Z})$,

$$\sup_{x\in\mathbb{R}^d} \sum_n |b_n(x)| \leq K(E, \Lambda_\epsilon)$$

and $\quad e^{-2\pi i x\cdot\gamma} = \sum_n b_n(x) e^{-2\pi i x_n \cdot \gamma}$ uniformly on $\Lambda_\epsilon$.

- Let h be entire on $\mathbb{R}^d$ with $e^{-\Omega(|x|)}$ decay,

$$h(0) = 1 \text{ and } \mathrm{supp}(\hat{h}) \subseteq \bar{B}(0, \epsilon).$$

**Theorem**

$\forall f \in C_b(\mathbb{R})$, *supp*$(\hat{f}) \subseteq \Lambda$,

$$\forall y \in \mathbb{R}^d, \quad f(y) = \sum f(x_n) b_n(y) h(x_n - y)$$

- Weighted sampling function $b_n(y)h(x_n - y)$ independent of $f \in C_b(\mathbb{R}^d)$, supp$(\hat{f}) \subseteq \Lambda$.

# Sampling formulas (2)

- The Nyquist condition, $2T\Omega \leq 1$, for sampling period $T$ and bandwidth $[-\Omega, \Omega]$, gives way to balayage (E,$\Lambda$), where $\Lambda$ is the bandwidth and the sampling set E is related to $\Lambda$ by balayage (E,$\Lambda$).
- Let $s \in C_b(\mathbb{R}^d)$, supp$(\hat{s}) \subseteq \Lambda$, a compact S-set - *sampling function* $s$.
- Let $A = \{a(n)\} \subseteq \mathbb{R}^d, n \in \mathbb{Z}$ and distinct points $a(n)$. Define

$$V_A = \{f \in C_b(\mathbb{R}^d) : \forall x \in \mathbb{R}^d, f(x) = \sum_n c_n(f)s(x-a(n)), \sum_n |c_n(f)| < \infty\}.$$

- Assume balayage (E,$\Lambda$), E $= \{x_n\} \subseteq \mathbb{R}^d$ separated.
- Define

$$V_E = \{f \in C_b(\mathbb{R}^d) : \forall x \in \mathbb{R}^d, f(x) = \sum_n c_n(f)s(x-x_n), \sum_n |c_n(f)| < \infty\}.$$

**Theorem**

$V = \cup_A V_A \subseteq V_E \subseteq C_b(\mathbb{R}^d)$. *Thus,*

$$\forall f \in V, \quad f(x) = \textstyle\sum_n c_n(f)s(x - x_n), \text{ uniformly on } \mathbb{R}^d.$$

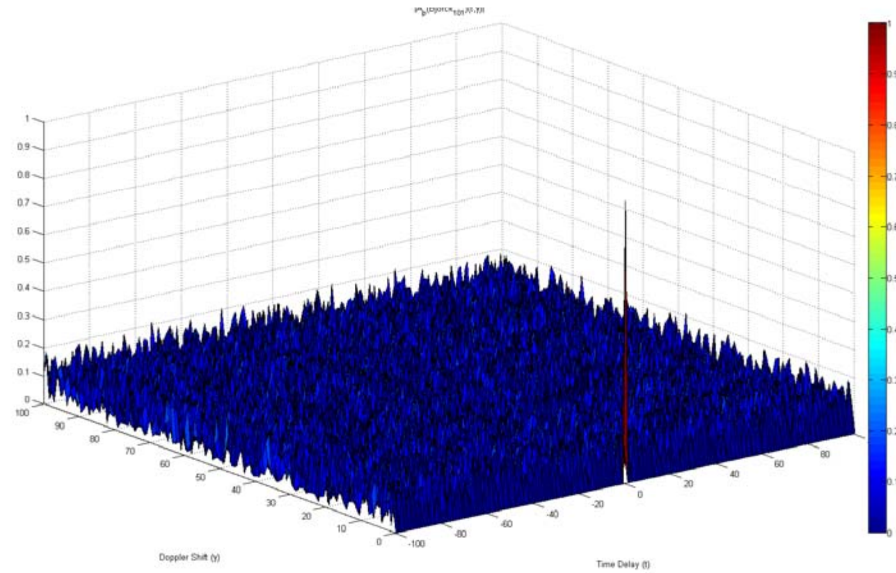
# Recent applications of FUNTFs

- Robust transmission of data over erasure channels such as the internet [Casazza, Goyal, Kelner, Kovačević]
- Multiple antenna code design for wireless communications [Hochwald, Marzetta,T. Richardson, Sweldens, Urbanke]
- Multiple description coding [Goyal, Heath, Kovačević, Strohmer,Vetterli]
- Quantum detection [Bölcskei, Eldar, Forney, Oppenheim, Kebo, B]
- Grassmannian "min-max" waveforms [Calderbank, Conway, Sloane, et al., Kolesar, B]

# Björck CAZAC Discrete Narrow-band Ambiguity Function

Let $u_p$ denote the Björck CAZAC sequence for prime $p$, and let $A(u_p)$ be its discrete narrow-band ambiguity function defined on $\mathbb{Z}_N \times \mathbb{Z}_N$.

**Theorem** (Benedetto-Woodworth) $\forall \epsilon > 0, \exists p_\epsilon$, a prime number which can be calculated, such that $\forall p \geq p_\epsilon$, $p$ prime, and $\forall (m,n) \in \mathbb{Z}_N \times \mathbb{Z}_N \backslash (0,0)$, $|A(u_p)(m,n)| < \epsilon$.




Norbert Wiener Center
for Harmonic Analysis and Applications

Classification

- Dimension reduction
- Finite frames and frame potential energy
- Frame potential energy classification algorithm
- Hyperspectral image processing

# Optimization problem: ideal class definition

- $Y = \{y_m\}_{m=1}^N \subseteq \mathbb{R}^d(K) \subseteq \mathbb{R}^N$
- Given $s$ classes $C_j, j = 1, \ldots, s$, defined in terms of a tolerance $\epsilon > 0$ and partition $\{P_j\}_{j=0}^s$ of $Y$, as $C_j = P_j \cap Y \subseteq B(z_j, \epsilon)$ $j = 1, \ldots, s$ for some $z_j \in \mathbb{R}^d(K)$.

# Optimization problem: FUNTF construction

- Point of view: Combine frame potential energy theorem, maximal separation criteria ($M_\delta$), and ideal class definition ($C_\epsilon$).
- Paradigm: Given $Y$, $s$, $M_\delta$, and $C_\epsilon$. Construct a FUNTF $\{\Psi_j\}_{j=1}^s$ such that

$$\forall j = 1, \ldots, s, \quad |\langle \Psi_j, Y \cap P_j \rangle| \geq R(\epsilon, \delta)$$

and

$$\forall j \neq k, \quad |\langle \Psi_k, Y \cap P_j \rangle| \leq s(\epsilon, \delta),$$

where $s(\epsilon, \delta) < R(\epsilon, \delta)$.

# Frame coefficient images

- Given $\Psi = \{\Psi_n\}_{n=1}^{s} \subseteq \mathbb{R}^d = \mathbb{R}^d(K) \subseteq \mathbb{R}^N$ and $m \in \{1, \ldots, N\}$. Consider the set of frame decompositions

$$\forall y_m \in \mathbb{R}^d, m = 1, \ldots, N, \quad y_m = \sum_{n=1}^{s} c_{m,n}^{\alpha} \Psi_n, \quad \text{indexed by } \alpha \in \mathbb{R}.$$

- For each $m \in \{1, \ldots, N\}$ choose an $\ell^1$ sparse decomposition

$$y_m = \sum_{n=1}^{s} c_{m,n}^{\alpha(m)} \Psi_n$$

defined by the inequality,

$$\forall \alpha, \quad \sum_{n=1}^{s} |c_{m,n}^{\alpha(m)}| \leq \sum_{n=1}^{s} |c_{m,n}^{\alpha}|.$$

# Optimization problem: FUNTF construction

Combine maximal separation with frame potential to construct a pseudo-FUNTF $\Psi = \{\psi_k\}_{k=1}^{s}$ by solving the minimization problem:

$$\sup \left\{ \min\{|p(\theta_k) - p(\theta_n)| : k \neq n\} : \{\theta_j\} \in \{\arg\min_{\Phi} TFP(\Phi)\right\}, \quad (1)$$

where $\Phi = \{\phi_k\}_{k=1}^{s}$.

- (1) is solved using a new, fast gradient descent method for products of spheres.
- Nate Strawn created the method and developed new geometric ideas for such computation.

Norbert Wiener Center for Harmonic Analysis and Applications

# Optimization problem: FUNTF construction

Combine frame potential with "$\ell^1$-energy" to construct a FUNTF $\Psi = \{\psi_k\}_{k=1}^s$ by solving a minimization problem of the following type:

$$\min\{TFP(\Theta) + P(Y, \Theta) : \Theta \in S^{d-1} \times \cdots \times S^{d-1}\},$$

where

$$P(Y, \Theta) = \sum_{n=1}^{N} \sum_{k=1}^{s} |\langle y_n, \theta_k \rangle| = \sum_{k=1}^{s} p(\theta_k).$$

Remark. a. Minimization of $P$ is convex optimization of $\ell^1$-energy of $Y$ for a given frame.
b. By Candes and Tao (2005), under suitable conditions, this can yield a frame $\Psi$ with a sparse set of coefficients $\{\langle y_n, \psi_k \rangle\}$. We do not proceed this way to obtain sparsity.

# Frame coefficient images (continued)

- For each $m \in \{1, \ldots, N\}$ choose an $\ell^1$ sparse decomposition

$$y_m = \sum_{n=1}^{s} c_{m,n}^{\alpha(m)} \Psi_n$$

defined by the inequality,

$$\forall\, \alpha, \quad \sum_{n=1}^{s} |c_{m,n}^{\alpha(m)}| \leq \sum_{n=1}^{s} |c_{m,n}^{\alpha}|.$$

- There is $\ell^0$ theory.

Norbert Wiener Center for Harmonic Analysis and Applications

# Frame coefficient images

Given $\Psi = \{\Psi_n\}_{n=1}^{s}$ and $m \in \{1, \ldots, N\}$. Consider the set of frame decompositions

$$y_m = \sum_{n=1}^{s} c_{m,n}^{\alpha} \Psi_n, \quad \text{indexed by } \alpha \in \mathbb{R}.$$

- If $\Psi$ is a FUNTF then $\alpha = 0$ designates the canonical dual, i.e.,

$$c_{m,n}^{0} = \frac{d}{s} \langle y_m, \Psi_n \rangle.$$

# Frame force and potential energy

$$F : S^{d-1} \times S^{d-1} \setminus D \longrightarrow \mathbb{R}^d$$

$$P : S^{d-1} \times S^{d-1} \setminus D \longrightarrow \mathbb{R},$$

where $P(a, b) = p(\|a - b\|), \qquad p'(x) = -xf(x)$

- Coulomb force

$$CF(a, b) = (a - b)/\|a - b\|^3, \quad f(x) = 1/x^3$$

- Frame force

$$FF(a, b) = \langle a, b\rangle(a - b), \quad f(x) = 1 - x^2/2$$

- Total potential energy for the frame force

$$TFP(\{x_n\}) = \sum_{m=1}^{N} \sum_{n=1}^{N} |\langle x_m, x_n\rangle|^2$$

# Characterization of FUNTFs

**Theorem**

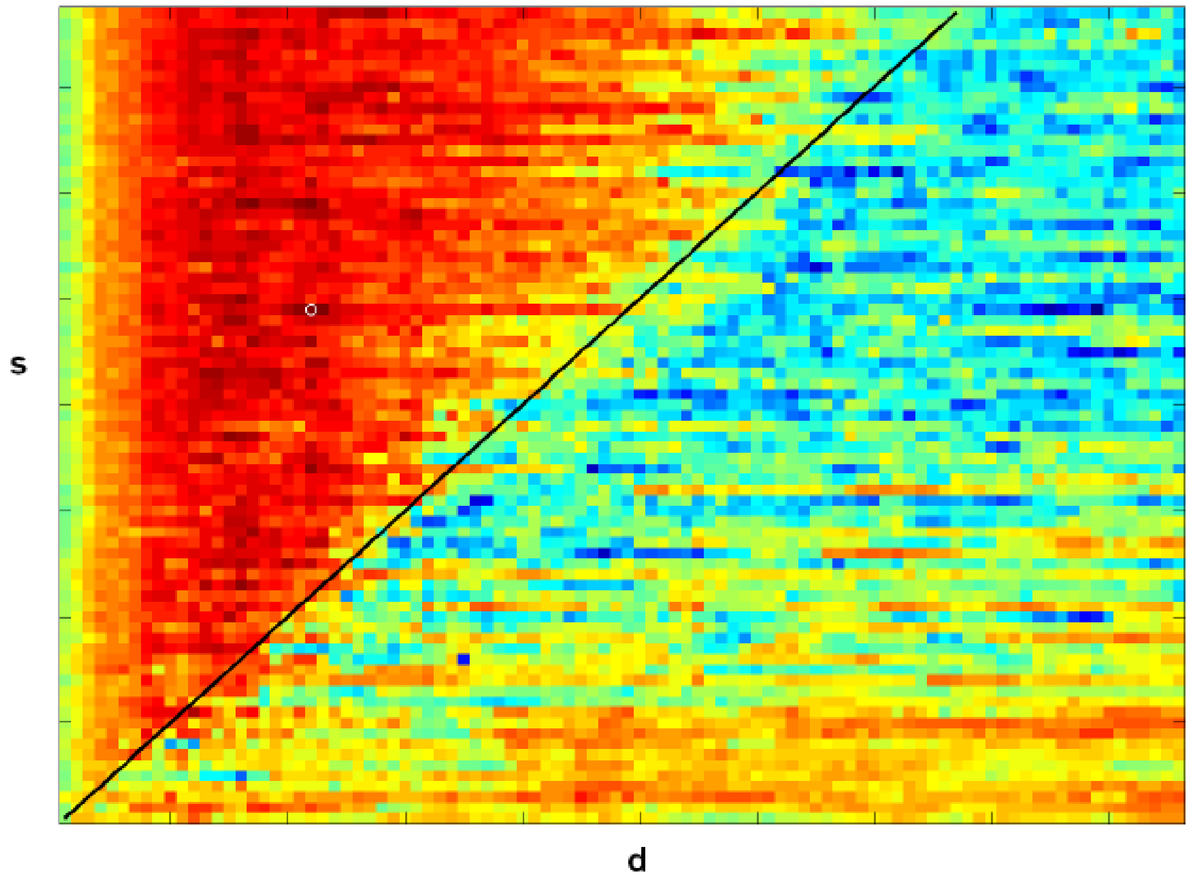
Let $N \leq d$. The minimum value of *TFP*, for the frame force and $N$ variables, is $N$; and the *minimizers* are precisely the orthonormal sets of $N$ elements for $\mathbb{R}^d$.

Let $N \geq d$. The minimum value of *TFP*, for the frame force and $N$ variables, is $N^2/d$; and the *minimizers* are precisely the FUNTFs of $N$ elements for $\mathbb{R}^d$.

**Problem**

Find FUNTFs analytically, effectively, computationally.

# Overview of Classification Results



# That’s all folks!

# Lower frame bounds

- Let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be a compact S-set, and assume balayage (E, $\Lambda$) where E $= \{x_n\}$ is separated.

1. $\forall F \in L^2(\Lambda),\ \Lambda$ convex,

$$\sqrt{A}\ \frac{\int_\Lambda |F(\gamma)+F(2\gamma)+F(3\gamma)|^2\ d\gamma}{(\int_\Lambda |F(\gamma)|^2\ d\gamma)^{1/2}}$$

$$\leq (\sum |\check{F}(x_n)|^2)^{1/2} + \tfrac{1}{2}(\sum |\check{F}(\tfrac{1}{2}x_n)|^2)^{1/2} + \tfrac{1}{3}(\sum |\check{F}(\tfrac{1}{3}x_n)|^2)^{1/2}.$$

2. Given positive $G \in L^2(\Lambda)$. Then $\forall F \in L^2(\Lambda)$,

$$\sqrt{A}\ \frac{\int_\Lambda |F(\gamma)|^2 G(\gamma)\ d\gamma}{(\int_\Lambda |F(\gamma)|^2\ d\gamma)^{\frac{1}{2}}} \leq (\sum |(FG)\check{}(x_n)|^2)^{1/2}.$$

Norbert Wiener Center for Harmonic Analysis and Applications

- Consider the data points $X$ as the nodes of a graph.
- Define a metric $\rho : X \times X \longrightarrow \mathbb{R}^+$, e.g., $\rho(x_m, x_n) = \|x_m - x_n\|$ is the Euclidean distance.
- Choose $q \in \mathbb{N}$.
- For each $x_i$ choose the $q$ nodes $x_n$ closest to $x_i$ in the metric $\rho$, and place an edge between $x_i$ and each of these nodes.
- This defines $N'(x_i)$, viz.,
  $N'(x_i) = \{x \in X \; : \; \exists$ an edge between $x$ and $x_i.\}$.
- To define the weights on the edges, we compute:

$$W = \operatorname*{argmin}_{\widetilde{W}} \left| x_i - \sum_{j \in N'(x_i)} \widetilde{W}(x_i, x_j) x_j \right|^2 .$$

- Set $K = (I - W)(I - W^{\mathsf{T}})$ and diagonalize $K$.
- $K$ is symmetric and positive semi–definite.

Norbert Wiener Center for Harmonic Analysis and Applications

# DFT FUNTFs

- $N \times d$ submatrices of the $N \times N$ DFT matrix are FUNTFs for $\mathbb{C}^d$. These play a major role in finite frame $\Sigma\Delta$-quantization.

$$N = 8,\ d = 5 \qquad \frac{1}{\sqrt{5}} \begin{bmatrix} * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \end{bmatrix}$$

$$x_m = \frac{1}{5}\left(e^{2\pi i \frac{m}{8}}, e^{2\pi i m \frac{2}{8}}, e^{2\pi i m \frac{5}{8}}, e^{2\pi i m \frac{6}{8}}, e^{2\pi i m \frac{7}{8}}\right)$$

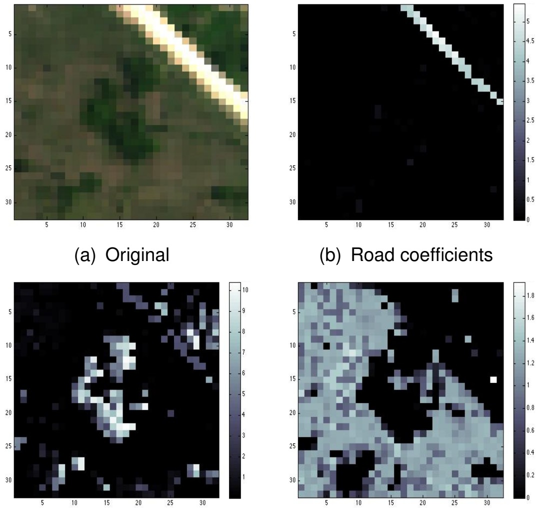
$$m = 1, \ldots, 8.$$

- Sigma-Delta Super Audio CDs - but not all authorities are fans.

# The geometry of finite tight frames

- We saw the vertices of platonic solids are FUNTFs.
- However, points that constitute FUNTFs do not have to be equidistributed, e.g., ONBs and Grassmanian frames.
- FUNTFs can be characterized as minimizers of a frame potential function (with Fickus) analogous to Coulomb's Law.
- Frame potential energy optimization has basic applications dealing with classification problems for hyperspectral and multi-spectral (biomedical) image data.

# Frame coefficients



(a) Original

(b) Road coefficients

(c) Tree coefficients

(d) Dirt/grass coefficients

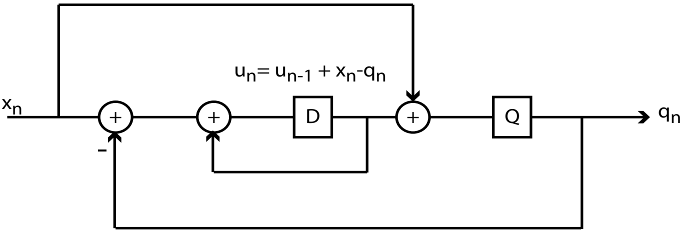
Norbert Wiener Center for Harmonic Analysis and Applications

# Quantization Methods

# SIGMA-DELTA QUANTIZATION

Given $u_0$ and $\{x_n\}_{n=1}$

$u_n = u_{n-1} + x_n - q_n$
$q_n = Q(u_{n-1} + x_n)$




First Order $\Sigma\Delta$

# A quantization problem

**Qualitative Problem** Obtain *digital* representations for class $X$, suitable for storage, transmission, recovery.
**Quantitative Problem** Find dictionary $\{e_n\} \subseteq X$:

1. Sampling [continuous range $\mathbb{K}$ is not digital]

$$\forall x \in X, \quad x = \sum x_n e_n, \quad x_n \in \mathbb{K}.$$

2. Quantization. Construct finite alphabet $\mathcal{A}$ and

$$Q: \ X \to \{\sum q_n e_n : \ q_n \in \mathcal{A} \subseteq \mathbb{K}\}$$

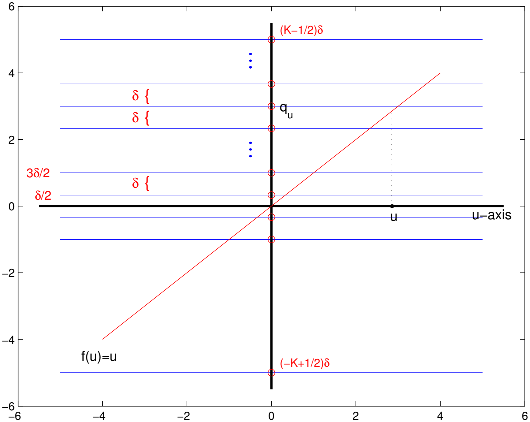
such that $|x_n - q_n|$ and/or $\|x - Qx\|$ small.

## Methods

Fine quantization, e.g., PCM. Take $q_n \in \mathcal{A}$ close to given $x_n$. Reasonable in 16-bit (65,536 levels)digital audio.
Coarse quantization, e.g., $\Sigma\Delta$. Use fewer bits to exploit redundancy.
SRQP

# Quantization

$$\mathcal{A}_K^\delta = \{(-K+1/2)\delta, (-K+3/2)\delta, \ldots, (-1/2)\delta, (1/2)\delta, \ldots, (K-1/2)\delta\}$$






$$Q(u) = \arg\min\{|u-q| : q \in \mathcal{A}_K^\delta\} = q_u$$

# PCM and first order Sigma-Delta

Let $x \in \mathbb{C}^d$, $\{e_n\}_{n=1}^N$ be a frame for $\mathbb{C}^d$.

- **PCM:** $\forall n = 1, \ldots, N, \quad q_n = Q_\delta(\langle x, e_n \rangle)$,
- **First Order Sigma-Delta:** Let $p$ be a permutation of $\{1, \ldots, N\}$. First Order Sigma-Delta quantization generates quantized sequence $\{q_n\}_{n=1}^N$ by the iteration

$$\begin{aligned} q_n &= Q_\delta(u_{n-1} + \langle x, e_{p(n)} \rangle), \\ u_n &= u_{n-1} + \langle x, e_{p(n)} \rangle - q_n, \end{aligned}$$

for $n = 1, \ldots, N$, with an initial condition $u_0$.

In either case, the quantized estimate is

$$\widetilde{x} = \frac{d}{N} \sum_{n=1}^{N} q_n e_n = \frac{d}{N} L^* q$$

# PCM

Replace $x_n \leftrightarrow q_n = \arg\{\min |x_n - q| : q \in \mathcal{A}_K^\delta\}$. Then

$$(PCM) \qquad \tilde{x} = \frac{d}{N}\sum_{n=1}^{N} q_n e_n$$

satisfies

$$\|x - \tilde{x}\| \leq \frac{d}{N}\|\sum_{n=1}^{N}(x_n - q_n)e_n\| \leq \frac{d}{N}\frac{\delta}{2}\sum_{n=1}^{N}\|e_n\| = \frac{d}{2}\delta.$$

Not good!

**Bennett's white noise assumption**

Assume that $(\eta_n) = (x_n - q_n)$ is a sequence of independent, identically distributed random variables with mean 0 and variance $\frac{\delta^2}{12}$. Then the mean square error (MSE) satisfies

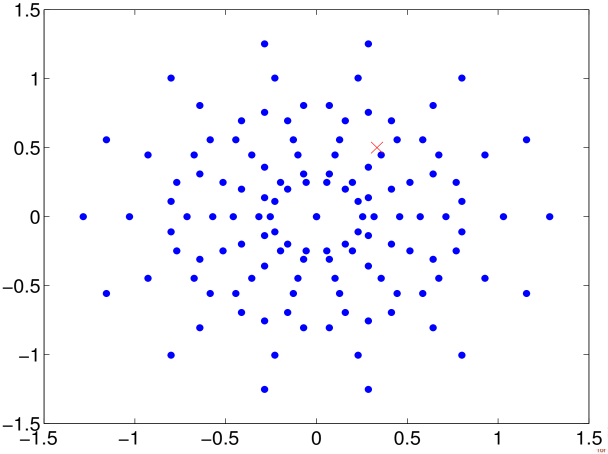
$$\text{MSE} = E\|x - \tilde{x}\|^2 \leq \frac{d}{12A}\,\delta^2 = \frac{(d\delta)^2}{12N}$$

# $\mathcal{A}_1^2 = \{-1, 1\}$ and $E_7$

Let $x = (\frac{1}{3}, \frac{1}{2})$, $E_7 = \{(\cos(\frac{2n\pi}{7}), \sin(\frac{2n\pi}{7}))\}_{n=1}^{7}$. Consider quantizers with $\mathcal{A} = \{-1, 1\}$.

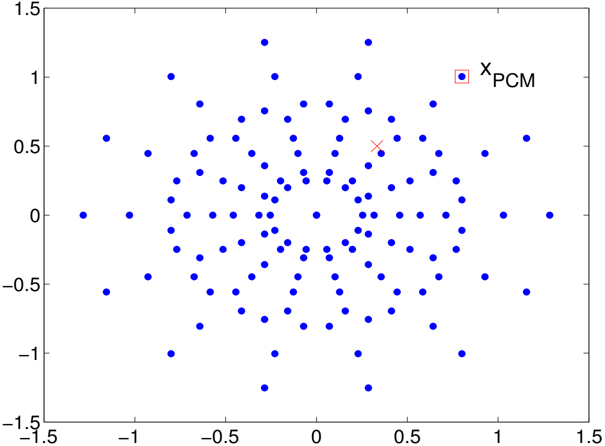
# $\mathcal{A}_1^2 = \{-1, 1\}$ and $E_7$

Let $x = (\frac{1}{3}, \frac{1}{2})$, $E_7 = \{(\cos(\frac{2n\pi}{7}), \sin(\frac{2n\pi}{7}))\}_{n=1}^{7}$. Consider quantizers with $\mathcal{A} = \{-1, 1\}$.



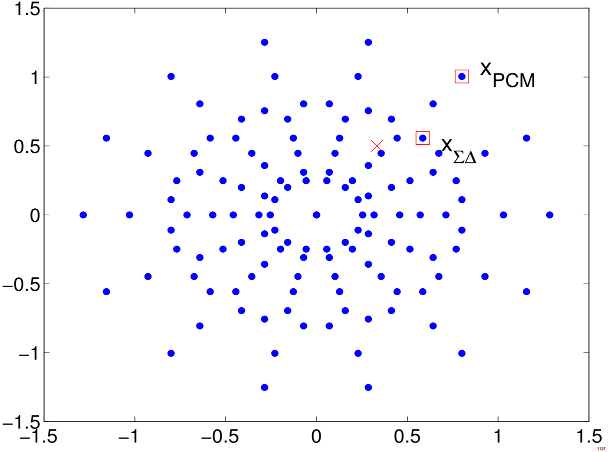
# $\mathcal{A}_1^2 = \{-1, 1\}$ and $E_7$

Let $x = (\frac{1}{3}, \frac{1}{2})$, $E_7 = \{(\cos(\frac{2n\pi}{7}), \sin(\frac{2n\pi}{7}))\}_{n=1}^{7}$. Consider quantizers with $\mathcal{A} = \{-1, 1\}$.



## $\mathcal{A}_1^2 = \{-1, 1\}$ and $E_7$

Let $x = (\frac{1}{3}, \frac{1}{2})$, $E_7 = \{(\cos(\frac{2n\pi}{7}), \sin(\frac{2n\pi}{7}))\}_{n=1}^{7}$. Consider quantizers with $\mathcal{A} = \{-1, 1\}$.



# Sigma-Delta quantization – background

- History from 1950s.
- Treatises of Candy, Temes (1992) and Norsworthy, Schreier, Temes (1997).
- PCM for finite frames and $\Sigma\Delta$ for $PW_{\Omega}$:
  Bölcskei, Daubechies, DeVore, Goyal, Güntürk, Kovačevič, Thao, Vetterli.
- Combination of $\Sigma\Delta$ and finite frames:
  Powell, Yılmaz, and B.
- Subsequent work based on this $\Sigma\Delta$ finite frame theory:
  Bodman and Paulsen; Boufounos and Oppenheim; Jimenez and Yang Wang; Lammers, Powell, and Yılmaz.
- Genuinely apply it.

# $\Sigma\Delta$ quantizers for finite frames

Let $F = \{e_n\}_{n=1}^N$ be a frame for $\mathbb{R}^d$, $x \in \mathbb{R}^d$.

**Define** $x_n = \langle x, e_n \rangle$.

**Fix the ordering** $p$, a permutation of $\{1, 2, \ldots, N\}$.

**Quantizer alphabet** $\mathcal{A}_K^\delta$

**Quantizer function** $Q(u) = \arg\{\min |u - q| : q \in \mathcal{A}_K^\delta\}$

Define the *first-order* $\Sigma\Delta$ *quantizer* with ordering $p$ and with the quantizer alphabet $\mathcal{A}_K^\delta$ by means of the following recursion.

$$\begin{aligned} u_n - u_{n-1} &= x_{p(n)} - q_n \\ q_n &= Q(u_{n-1} + x_{p(n)}) \end{aligned}$$

where $u_0 = 0$ and $n = 1, 2, \ldots, N$.

# Stability

The following stability result is used to prove error estimates.

**Proposition**

If the frame coefficients $\{x_n\}_{n=1}^N$ satisfy

$$|x_n| \le (K - 1/2)\delta, \quad n = 1, \cdots, N,$$

then the state sequence $\{u_n\}_{n=0}^N$ generated by the first-order $\Sigma\Delta$ quantizer with alphabet $\mathcal{A}_K^\delta$ satisfies $|u_n| \le \delta/2, n = 1, \cdots, N.$

- The first-order $\Sigma\Delta$ scheme is equivalent to

$$u_n = \sum_{j=1}^{n} x_{p(j)} - \sum_{j=1}^{n} q_j, \quad n = 1, \cdots, N.$$

- Stability results lead to tiling problems for higher order schemes.

# Error estimate

## Definition

Let $F = \{e_n\}_{n=1}^N$ be a frame for $\mathbb{R}^d$, and let $p$ be a permutation of $\{1, 2, \ldots, N\}$. The *variation* $\sigma(F, p)$ is

$$\sigma(F, p) = \sum_{n=1}^{N-1} \|e_{p(n)} - e_{p(n+1)}\|.$$

# Error estimate

**Theorem**

Let $F = \{e_n\}_{n=1}^{N}$ be an $A$-FUNTF for $\mathbb{R}^d$. The approximation

$$\tilde{x} = \frac{d}{N} \sum_{n=1}^{N} q_n e_{p(n)}$$

generated by the first-order $\Sigma\Delta$ quantizer with ordering $p$ and with the quantizer alphabet $\mathcal{A}_K^{\delta}$ satisfies

$$\|x - \tilde{x}\| \leq \frac{(\sigma(F,p)+1)d}{N} \frac{\delta}{2}.$$

# Harmonic frames

Zimmermann and Goyal, Kelner, Kovačević, Thao, Vetterli.

**Definition**

$H = \mathbb{C}^d$. An *harmonic frame* $\{e_n\}_{n=1}^N$ for $H$ is defined by the rows of the Bessel map $L$ which is the complex $N$-DFT $N \times d$ matrix with $N - d$ columns removed.

$H = \mathbb{R}^d$, $d$ even. The harmonic frame $\{e_n\}_{n=1}^N$ is defined by the Bessel map $L$ which is the $N \times d$ matrix whose $n$th row is

$$e_n^N = \sqrt{\frac{2}{d}}\left(\cos(\frac{2\pi n}{N}), \sin(\frac{2\pi n}{N}), \ldots, \cos(\frac{2\pi(d/2)n}{N}), \sin(\frac{2\pi(d/2)n}{N})\right).$$

- Harmonic frames are FUNTFs.
- Let $E_N$ be the harmonic frame for $\mathbb{R}^d$ and let $p_N$ be the identity permutation. Then

$$\forall N, \ \sigma(E_N, p_N) \leq \pi d(d+1).$$

# Error estimate for harmonic frames

**Theorem**

Let $E_N$ be the harmonic frame for $\mathbb{R}^d$ with frame bound $N/d$. Consider $x \in \mathbb{R}^d$, $\|x\| \le 1$, and suppose the approximation $\tilde{x}$ of $x$ is generated by a first-order $\Sigma\Delta$ quantizer as before. Then

$$\|x - \tilde{x}\| \le \frac{d^2(d+1)+d}{N} \frac{\delta}{2}.$$

- Hence, for harmonic frames (and all those with bounded variation),

$$\mathrm{MSE}_{\Sigma\Delta} \le \frac{C_d}{N^2} \delta^2.$$

- This bound is clearly superior asymptotically to

$$\mathrm{MSE}_{\mathrm{PCM}} = \frac{(d\delta)^2}{12N}.$$

# $\Sigma\Delta$ and "optimal" PCM

**Theorem**

The first order $\Sigma\Delta$ scheme achieves the asymptotically optimal $\mathrm{MSE}_{\mathrm{PCM}}$ for harmonic frames.

The digital encoding

$$\mathrm{MSE}_{\mathrm{PCM}} = \frac{(d\delta)^2}{12N}$$

in PCM format leaves open the possibility that decoding (consistent nonlinear reconstruction, with additional numerical complexity this entails) could lead to

$$\text{“}\mathrm{MSE}_{\mathrm{PCM}}^{\mathrm{opt}}\text{”} \ll O\left(\frac{1}{N}\right).$$

Goyal, Vetterli, Thao (1998) proved

$$\text{“}\mathrm{MSE}_{\mathrm{PCM}}^{\mathrm{opt}}\text{”} \sim \frac{\tilde{C}_d}{N^2}\delta^2.$$

# A comparison of Σ-Δ and PCM

# Comparison of 1-bit PCM and 1-bit $\Sigma\Delta$

Let $x \in \mathbb{C}^d$, $||x|| \leq 1$.

**Definition**

- $q_{PCM}(x)$ is the sequence to which $x$ is mapped by PCM.
- $q_{\Sigma\Delta}(x)$ is the sequence to which $x$ is mapped by $\Sigma\Delta$.
- $$\mathrm{err}_{PCM}(x) = ||x - \frac{d}{N} L^* q_{PCM}(x)||$$
$$\mathrm{err}_{\Sigma\Delta}(x) = ||x - \frac{d}{N} L^* q_{\Sigma\Delta}(x)||$$

Fickus question: We shall analyze to what extent $\mathrm{err}_{\Sigma\Delta}(x) < \mathrm{err}_{PCM}(x)$ beyond our results with Powell and Yilmaz.

# PCM and first order Sigma-Delta

Let $x \in \mathbb{C}^d$,
Let $F = \{e_n\}_{n=1}^N$ be a FUNTF for $\mathbb{C}^d$ with the analysis matrix $L$.

## Definition

- $q_{PCM}(x, F, b)$ is the quantized sequence given by $b$-bit PCM,
- $q_{\Sigma\Delta}(x, F, b)$ is the quantized sequence given by $b$-bit Sigma-Delta.
- $$\mathtt{err}_{PCM}(x, F, b) = \|x - \frac{d}{N} L^* q_{PCM}(x)\|,$$ $$\mathtt{err}_{\Sigma\Delta}(x, F, b) = \|x - \frac{d}{N} L^* q_{\Sigma\Delta}(x)\|.$$

Norbert Wiener Center for Harmonic Analysis and Applications

# Comparison of 1-bit PCM and 1-bit $\Sigma\Delta$

**Definition**

A function $e : [a, b] \to \mathbb{C}^d$ is of *bounded variation (BV)* if there is a $K > 0$ such that for every $a \leq t_1 < t_2 < \cdots < t_N \leq b$,

$$\sum_{n=1}^{N-1} \|e(t_n) - e(t_{n+1})\| \leq K.$$

The smallest such $K$ is denoted by $|e|_{BV}$, and defines a seminorm for the space of BV functions.

# Comparison of 1-bit PCM and 1-bit Sigma-Delta

**Theorem 1**

Let $x \in \mathbb{C}^d$ satisfy $0 < \|x\| \leq 1$, and let $F = \{e_n\}_{n=1}^N$ be a FUNTF for $\mathbb{C}^d$. Then, the 1-bit PCM error satisfies

$$\mathtt{err}_{PCM}(x, F, 1) \geq \alpha_F + 1 - \|x\|$$

where

$$\alpha_F := \inf_{\|x\|=1} \frac{d}{N} \sum_{n=1}^{N} (|Re(\langle x, e_n \rangle)| + |Im(\langle x, e_n \rangle)|) - 1 \geq 0.$$

# Comparison of 1-bit PCM and 1-bit Sigma-Delta

**Theorem 2**

Let $\{F_N = \{e_n^N\}_{n=1}^N\}$ be a family of FUNTFs for $\mathbb{C}^d$. Then,

$$\forall \varepsilon > 0,\ \exists N_0 > 0,\ \ \textit{such that}\ \ \forall N \geq N_0\ \ \textit{and}\ \ \forall 0 < \|x\| \leq 1 - \varepsilon$$

$$\mathtt{err}_{\Sigma\Delta}(x, F_N, 1) \leq \mathtt{err}_{PCM}(x, F_N, 1).$$

Numerical experiments suggest that, we can choose $N$ significantly smaller than $(M/\varepsilon)^{2d}$.

# Comparison of 1-bit PCM and 1-bit Sigma-Delta

If $\{\alpha_{F_N}\}$ is bounded below by a positive number, then we can improve Theorem 2:

**Theorem 3**

Let $\{F_N = \{e_n^N\}_{n=1}^N\}$ be a family of FUNTFs for $\mathbb{C}^d$ such that

$$\exists a > 0,\ \forall N,\ \alpha_{F_N} \geq a.$$

Then,

$$\exists N_0 > 0 \ \textit{such that} \ \forall N \geq N_0 \ \textit{and} \ \forall 0 < \|x\| \leq 1$$

$$\mathtt{err}_{\Sigma\Delta}(x, F_N, 1) \leq \mathtt{err}_{PCM}(x, F_N, 1).$$

# Comparison of 1-bit PCM and 1-bit Sigma-Delta

Below is a family $\{F_N\}$ of FUNTFs where $\{\alpha_{F_N}\}$ is bounded below by a positive constant. Harmonic frames are examples to such families.

**Theorem 4**

Let $e : [0,1] \to \{x \in \mathbb{C}^d : \|x\| = 1\}$ be continuous function of bounded variation such that $F_N = \{e(n/N)\}_{n=1}^N$ is a FUNTF for $\mathbb{C}^d$ for every $N$. Then,

$$\exists N_0 > 0 \textit{ such that } \forall N \geq N_0 \textit{ and } \forall 0 < \|x\| \leq 1$$

$$\mathtt{err}_{\Sigma\Delta}(x, F_N, 1) \leq \mathtt{err}_{PCM}(x, F_N, 1).$$

One can show that $\alpha := \lim_{N\to\infty} \alpha_{F_N}$ is positive, and that

$$\alpha + 1 = d \inf_{\|x\|=1} \int_0^1 \left(|Re(\langle x, e(t)\rangle)| + |Im(\langle x, e(t)\rangle)|\right) dt.$$

# Comparison of 1-bit PCM and 1-bit $\Sigma\Delta$

**Theorem**

Let $e : [0,1] \to \{x \in \mathbb{C}^d : \|x\| = 1\}$ be continuous function of bounded variation such that $F_N = (e(n/N))_{n=1}^N$ is a FUNTF for $\mathbb{C}^d$ for every $N$. Then,

$$\exists N_0 > 0 \textit{ such that } \forall N \geq N_0 \textit{ and } \forall 0 < \|x\| \leq 1$$

$$\mathtt{err}_{\Sigma\Delta}(x) \leq \mathtt{err}_{PCM}(x).$$

Moreover, a lower bound for $N_0$ is $d(1 + |e|_{BV})/(\sqrt{d} - 1)$.

# Comparison of 1-bit PCM and 1-bit $\Sigma\Delta$

**Example (Roots of unity frames for $\mathbb{R}^2$)**

$e_n^N = (\cos(2\pi n/N), \sin(2\pi n/N))$.

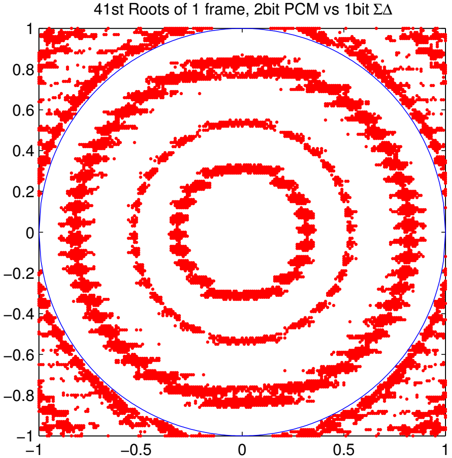
Here, $e(t) = (\cos(2\pi t), \sin(2\pi t))$,
$M = |e|_{BV} = 2\pi$, $lim \alpha_{F_N} = 2/\pi$.

**Example (Real Harmonic Frames for $\mathbb{R}^{2k}$)**

$e_n^N = \frac{1}{\sqrt{k}}(\cos(2\pi n/N), \sin(2\pi n/N), \dots, \cos(2\pi kn/N), \sin(2\pi kn/N))$.

In this case, $e(t) = \frac{1}{\sqrt{k}}(\cos(2\pi t), \sin(2\pi t), \dots, \cos(2\pi kt), \sin(2\pi kt))$,
$M = |e|_{BV} = 2\pi\sqrt{\frac{1}{d}\sum_{k=1}^{d} k^2}$.

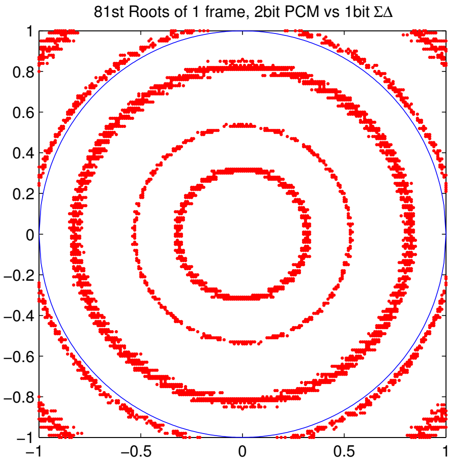
# Comparison of 2-bit PCM and 1-bit $\Sigma\Delta$




Red: $\mathtt{err}_{PCM}(x) < \mathtt{err}_{\Sigma\Delta}(x)$, Green: $\mathtt{err}_{PCM}(x) = \mathtt{err}_{\Sigma\Delta}(x)$.

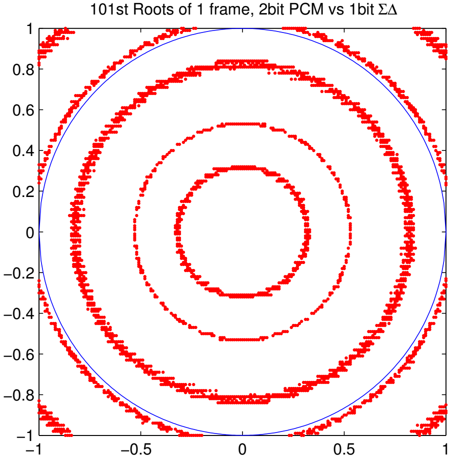
Norbert Wiener Center for Harmonic Analysis and Applications

# Comparison of 2-bit PCM and 1-bit $\Sigma\Delta$



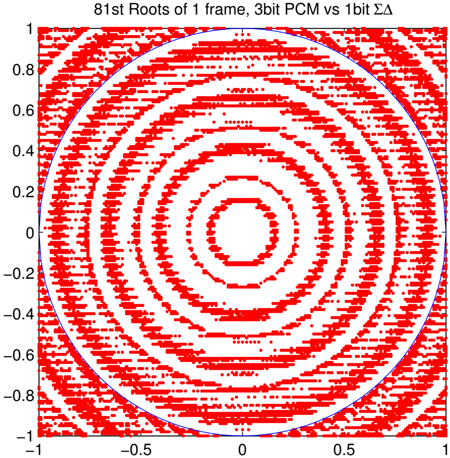

Red: $\mathtt{err}_{PCM}(x) < \mathtt{err}_{\Sigma\Delta}(x)$, Green: $\mathtt{err}_{PCM}(x) = \mathtt{err}_{\Sigma\Delta}(x)$.

# Comparison of 2-bit PCM and 1-bit $\Sigma\Delta$



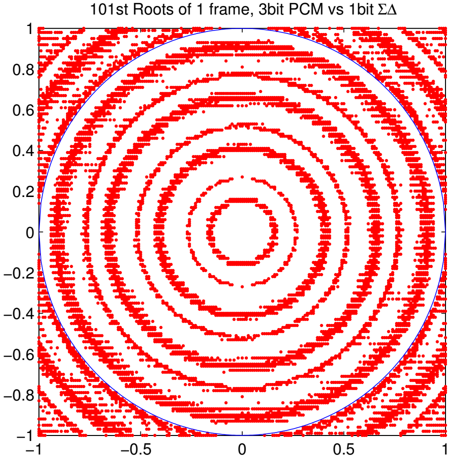

Red: $\mathtt{err}_{PCM}(x) < \mathtt{err}_{\Sigma\Delta}(x)$, Green: $\mathtt{err}_{PCM}(x) = \mathtt{err}_{\Sigma\Delta}(x)$.

# Comparison of 3-bit PCM and 1-bit $\Sigma\Delta$



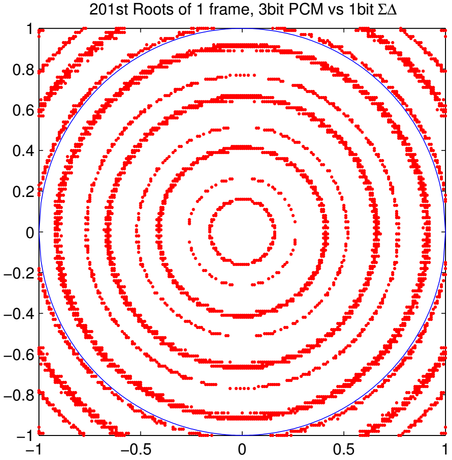

Red: $\mathtt{err}_{PCM}(x) < \mathtt{err}_{\Sigma\Delta}(x)$, Green: $\mathtt{err}_{PCM}(x) = \mathtt{err}_{\Sigma\Delta}(x)$.

# Comparison of 3-bit PCM and 1-bit ΣΔ



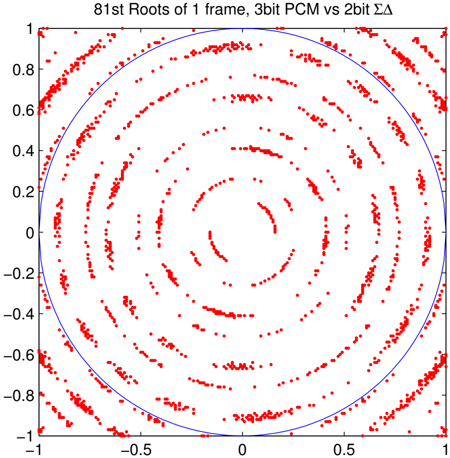

Red: $\mathtt{err}_{PCM}(x) < \mathtt{err}_{\Sigma\Delta}(x)$, Green: $\mathtt{err}_{PCM}(x) = \mathtt{err}_{\Sigma\Delta}(x)$.

# Comparison of 3-bit PCM and 1-bit $\Sigma\Delta$



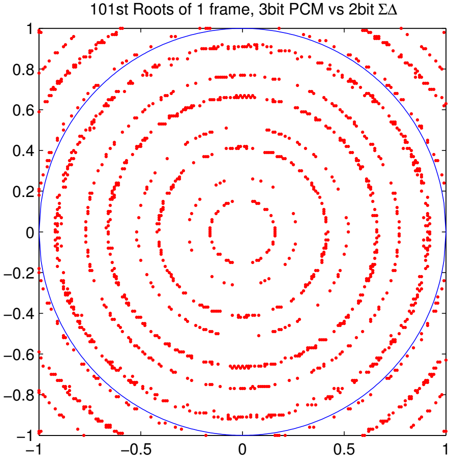

Red: $\mathtt{err}_{PCM}(x) < \mathtt{err}_{\Sigma\Delta}(x)$, Green: $\mathtt{err}_{PCM}(x) = \mathtt{err}_{\Sigma\Delta}(x)$.

# Comparison of 3-bit PCM and 2-bit ΣΔ



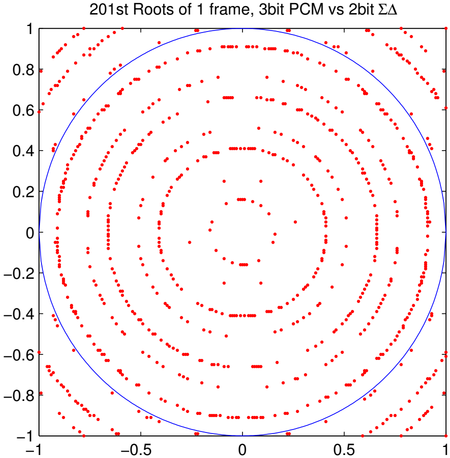

Red: $\mathtt{err}_{PCM}(x) < \mathtt{err}_{\Sigma\Delta}(x)$, Green: $\mathtt{err}_{PCM}(x) = \mathtt{err}_{\Sigma\Delta}(x)$.

# Comparison of 3-bit PCM and 2-bit ΣΔ




Red: $\mathtt{err}_{PCM}(x) < \mathtt{err}_{\Sigma\Delta}(x)$, Green: $\mathtt{err}_{PCM}(x) = \mathtt{err}_{\Sigma\Delta}(x)$.

# Comparison of 3-bit PCM and 2-bit ΣΔ




Red: $\mathtt{err}_{PCM}(x) < \mathtt{err}_{\Sigma\Delta}(x)$, Green: $\mathtt{err}_{PCM}(x) = \mathtt{err}_{\Sigma\Delta}(x)$.

# Complex ΣΔ - Alphabet

Let $K \in \mathbb{N}$ and $\delta > 0$. The *midrise* quantization alphabet is

$$\mathcal{A}_K^\delta = \left\{ \left( m + \frac{1}{2} \right) \delta + in\delta : m = -K, \ldots, K-1, \; n = -K, \ldots, K \right\}$$

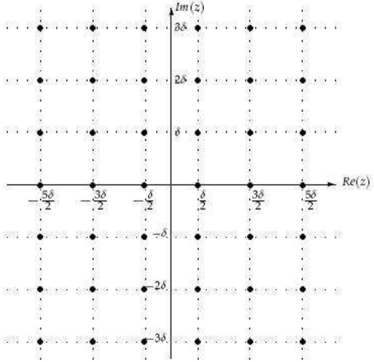



**Figure:** $\mathcal{A}_K^\delta$ for $K = 3\delta$.

# Alphabet

For $K > 0$ (we consider only $K = 1$)
and $b \geq 1$, an integer representing the number of bits,
let $\delta = 2K/(2^b - 1)$.

$$\mathcal{A}_\delta^K = \{(-K + m\delta) + i(-K + n\delta) : m, n = 0, \ldots, 2^b - 1\}.$$

The associated *scalar uniform quantizer* is

$$Q_\delta(u + iv) = \delta\left(\frac{1}{2} + \left\lfloor \frac{u}{\delta} \right\rfloor + i(\frac{1}{2} + \left\lfloor \frac{v}{\delta} \right\rfloor)\right).$$

In particular, for 1-bit case, $Q(u + iv) = sign(u) + isign(v)$

Norbert Wiener Center for Harmonic Analysis and Applications

# Complex $\Sigma\Delta$

The *scalar uniform quantizer* associated to $\mathcal{A}_K^\delta$ is

$$Q_\delta(a+ib) = \delta\left(\frac{1}{2} + \left\lfloor \frac{a}{\delta} \right\rfloor + i\left\lfloor \frac{b}{\delta} \right\rfloor\right),$$

where $\lfloor x \rfloor$ is the largest integer smaller than $x$.
For any $z = a + ib$ with $|a| \leq K$ and $|b| \leq K$, $Q$ satisfies

$$|z - Q_\delta(z)| \leq \min_{\zeta \in \mathcal{A}_K^\delta} |z - \zeta|.$$

Let $\{x_n\}_{n=1}^N \subseteq \mathbb{C}$ and let $p$ be a permutation of $\{1, \ldots, N\}$. Analogous to the real case, the first order $\Sigma\Delta$ quantization is defined by the iteration

$$\begin{aligned} u_n &= u_{n-1} + x_{p(n)} - q_n, \\ q_n &= Q_\delta(u_{n-1} + x_{p(n)}). \end{aligned}$$

# Complex $\Sigma\Delta$

The following theorem is analogous to BPY

**Theorem**

Let $F = \{e_n\}_{n=1}^N$ be a finite unit norm frame for $\mathbb{C}^d$, let $p$ be a permutation of $\{1, \dots, N\}$, let $|u_0| \leq \delta/2$, and let $x \in \mathbb{C}^d$ satisfy $\|x\| \leq (K - 1/2)\delta$. The $\Sigma\Delta$ approximation error $\|x - \widetilde{x}\|$ satisfies

$$\|x - \widetilde{x}\| \leq \sqrt{2}\|S^{-1}\|_{\mathrm{op}} \left( \sigma(F, p)\frac{\delta}{2} + |u_N| + |u_0| \right),$$

where $S^{-1}$ is the inverse frame operator. In particular, if $F$ is a FUNTF, then

$$\|x - \widetilde{x}\| \leq \sqrt{2}\frac{d}{N} \left( \sigma(F, p)\frac{\delta}{2} + |u_N| + |u_0| \right),$$

# Complex $\Sigma\Delta$

Let $\{F_N\}$ be a family of FUNTFs, and $p_N$ be a permutation of $\{1, \ldots, N\}$. Then the frame variation $\sigma(F_N, p_N)$ is a function of $N$. If $\sigma(F_N, p_N)$ is bounded, then

$$\|x - \widetilde{x}\| = \mathcal{O}(N^{-1}) \text{ as } N \to \infty.$$

Wang gives an upper bound for the frame variation of frames for $\mathbb{R}^d$, using the results from the Travelling Salesman Problem.

**Theorem YW**

Let $S = \{v_j\}_{j=1}^N \subseteq [-\frac{1}{2}, \frac{1}{2}]^d$ with $d \geq 3$. There exists a permutation $p$ of $\{1, \ldots, N\}$ such that

$$\sum_{j=1}^{N-1} \|v_{p(j)} - v_{p(j+1)}\| \leq 2\sqrt{d+3}N^{1-\frac{1}{d}} - 2\sqrt{d+3}.$$

# Complex $\Sigma\Delta$

**Theorem**

Let $F = \{e_n\}_{n=1}^N$ be a FUNTF for $\mathbb{R}^d$, $|u_0| \leq \delta/2$, and let $x \in \mathbb{R}^d$ satisfy $\|x\| \leq (K - 1/2)\delta$. Then, there exists a permutation $p$ of $\{1, 2, \ldots, N\}$ such that the approximation error $\|x - \widetilde{x}\|$ satisfies

$$\|x - \widetilde{x}\| \leq \sqrt{2}\delta d \left((1 - \sqrt{d+3})N^{-1} + \sqrt{d+3}N^{-\frac{1}{d}}\right)$$

This theorem guarantees that

$$\|x - \widetilde{x}\| \leq \mathcal{O}(N^{-\frac{1}{d}}) \text{ as } N \to \infty$$

for FUNTFs for $\mathbb{R}^d$.

Norbert Wiener Center for Harmonic Analysis and Applications

# Preprocessing for clutter mitigation

- Massive sensor data set → dimension reduction → sparse representation
- False targets caused by clutter inhibit data triage, waste vital resources, and degrade sparse representation algorithms
- View clutter mitigation as preprocessing step for ATR/ATE
- For active sensors, choose waveform to reduce clutter effects by limiting side lobe magnitude
  - improves concise data representation
  - supports dimensionality reduction processing

# Sparse coefficient sets for stable representation

- Opportunistic sensing systems can utilize large networks of diverse sensors
  - sensor quality may vary, e.g., low cost wireless sensors
  - massive amount of noisy sensor data
- Signal representations using sparse coefficient sets
  - compensate for hardware errors
  - ensure numerical stability
  - frame setting → frame dimension reduction

# Frame variation and $\Sigma\Delta$

- $F = \{e_j\}_{j=1}^N$ a FUNTF for $\mathbb{C}^d$
- $x \in \mathbb{C}^d$, $p$ a permutation of $\{1, ..., N\}$, $x_{p(n)} = \langle x, e_{p(n)} \rangle$,

$$x = \frac{d}{N} \sum_{n=1}^{N} x_{p(n)} e_{p(n)} \text{ and } \tilde{x} \equiv \frac{d}{N} \sum_{n=1}^{N} q_n e_{p(n)}$$

- Frame variation,

$$\sigma(F, p) = \sum_{n=1}^{N-1} \left\| e_{p(n)} - e_{p(n+1)} \right\|$$

- Transport $\Sigma\Delta$ FUNTF setting to coefficient sparse representation point of view.

# Summary

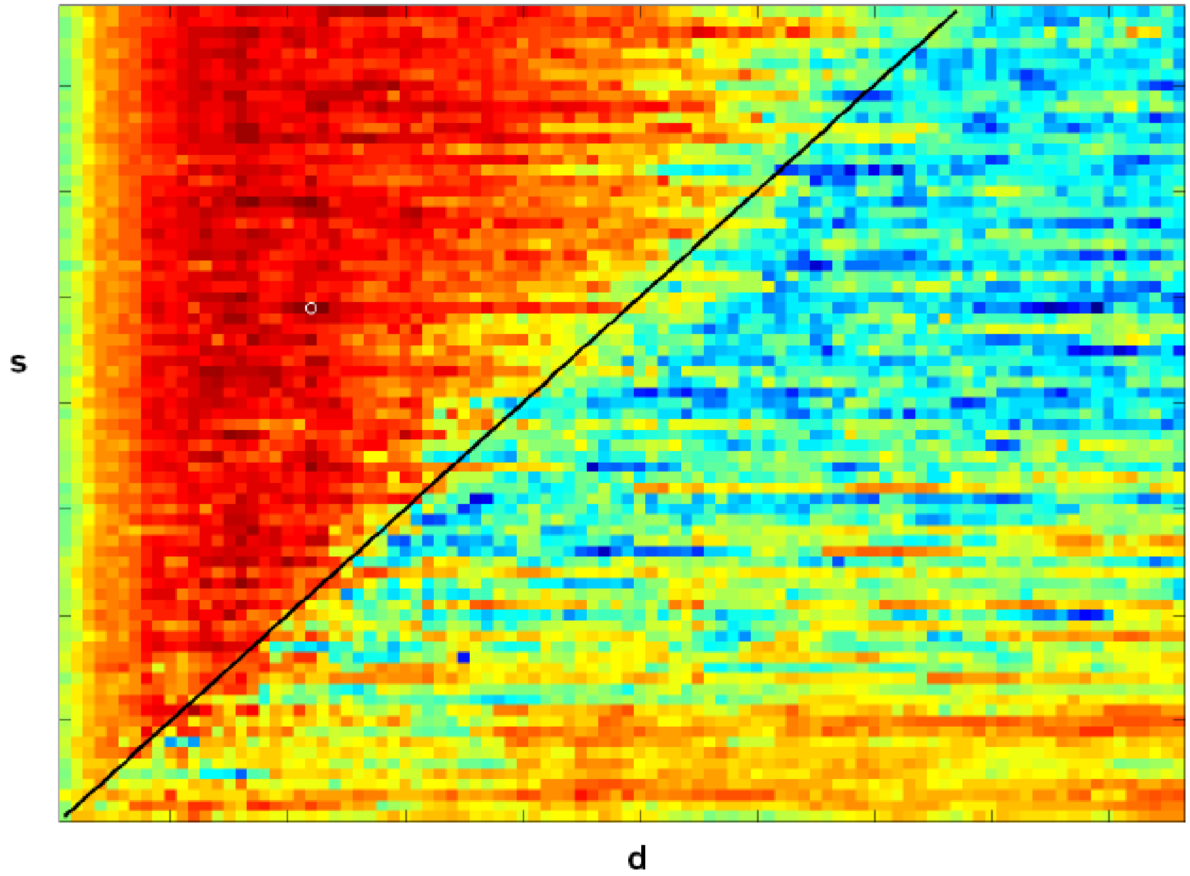
Given a signal x and a tolerance $r > 0$

- Define frames using Frame Potential Energy and SQP (or other optimization)
- Analyze Frame Variation in terms of our permutation algorithm
- Compute $\tilde{x}$ having separated coefficients taken from a fixed small and sparse alphabet
- Ensure that $\|x - \tilde{x}\| < r$.

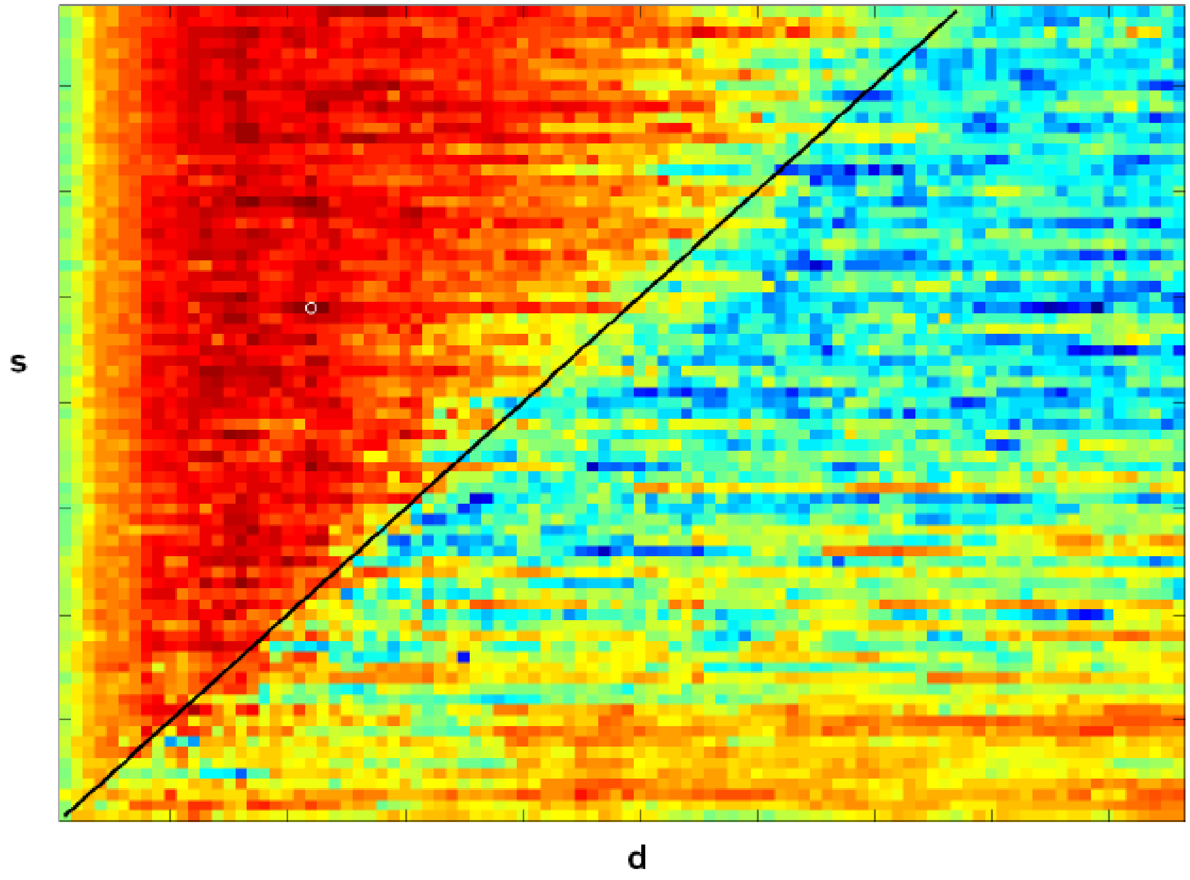
**Conclusion:** $\tilde{x}$ is a stable sparse coefficient approximant of x

Norbert Wiener Center
for Harmonic Analysis and Applications

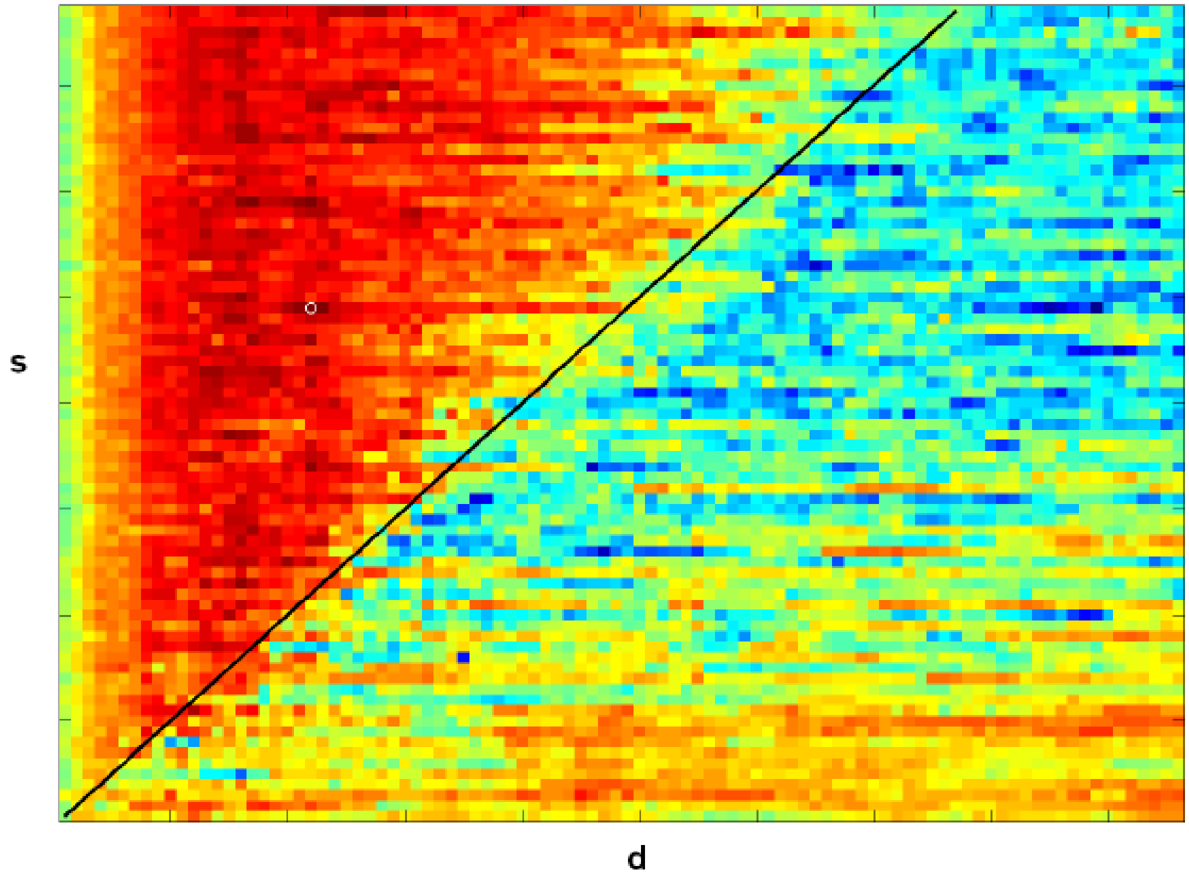
# Overview of Classification Results



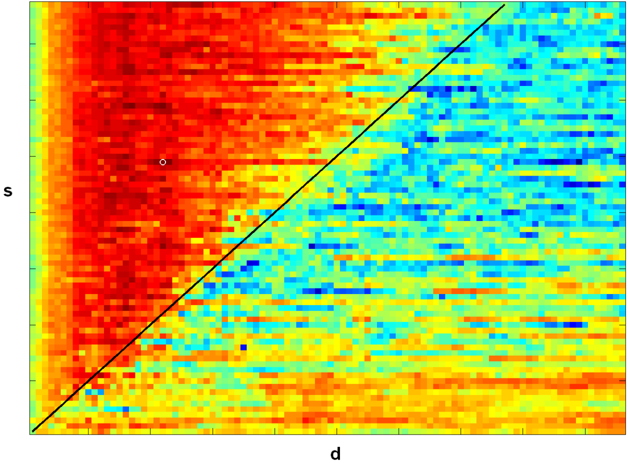
# Overview of Classification Results



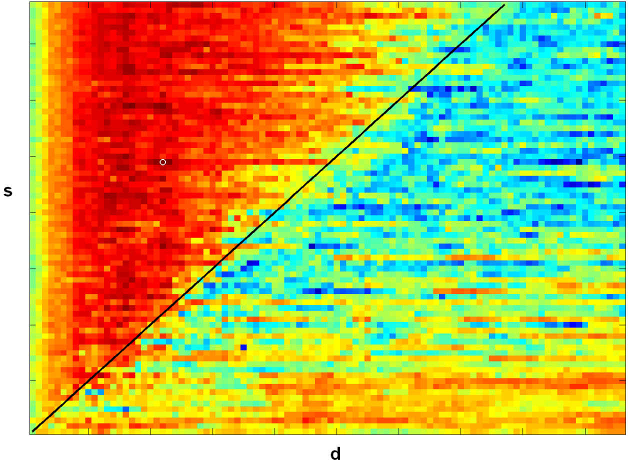
# Overview of Classification Results



# Overview of Classification Results



# Overview of Classification Results



# A vector-valued ambiguity function

# Outline



Norbert Wiener Center
for Harmonic Analysis and Applications

- Originally, our problem was to construct libraries of phase-coded waveforms $v$ parameterized by design variables, for communications and radar.
- A goal was to achieve diverse ambiguity function behavior of $v$ by defining new classes of quadratic phase and number theoretic perfect autocorrelation codes $u$ with which to define $v$.
- A realistic more general problem was to construct vector-valued waveforms $v$ in terms of vector-valued perfect autocorrelation codes $u$. Such codes are relevant in light of vector sensor and MIMO capabilities and modeling.
- Example: Discrete time data vector $u(k)$ for a $d$-element array,

$$k \longmapsto u(k) = (u_0(k), \ldots, u_{d-1}(k)) \in \mathbb{C}^d.$$

We can have $\mathbb{R}^N \to GL(d, \mathbb{C})$, or even more general.

- Establish the theory of vector-valued ambiguity functions to estimate $v$ in terms of ambiguity data.
- First, establish this estimation theory by defining the discrete periodic vector-valued ambiguity function in a natural way.
- Mathematically, this natural way is to formulate the discrete periodic vector-valued ambiguity function in terms of the Short Time Fourier Transform (STFT).

# STFT and ambiguity function

## Short time Fourier transform – STFT

- The narrow band cross-correlation ambiguity function of $v, w$ defined on $\mathbb{R}$ is

$$A(v,w)(t,\gamma) = \int_{\mathbb{R}} v(s+t)\overline{w(s)}e^{-2\pi i s\gamma}\,ds.$$

- $A(v,w)$ is the STFT of $v$ with window $w$.
- The *narrow band radar ambiguity function* $A(v)$ of $v$ on $\mathbb{R}$ is

$$A(v)(t,\gamma) = \int_{\mathbb{R}} v(s+t)\overline{v(s)}e^{-2\pi i s\gamma}\,ds$$

$$= e^{\pi i t\gamma}\int_{\mathbb{R}} v\left(s+\frac{t}{2}\right)\overline{v\left(s-\frac{t}{2}\right)}e^{-2\pi i s\gamma}\,ds, \text{ for } (t,\gamma)\in\mathbb{R}^2.$$

Norbert Wiener Center for Harmonic Analysis and Applications

# Goal

- Let $v$ be a phase coded waveform with $N$ lags defined by the code $u$.
- Let $u$ be $N$-periodic, and so $u : \mathbb{Z}_N \longrightarrow \mathbb{C}$, where $\mathbb{Z}_N$ is the additive group of integers modulo $N$.
- The *discrete periodic ambiguity function* $A_p(u) : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}$ is

$$A_p(u)(m,n) = \frac{1}{N} \sum_{k=0}^{N-1} u(m+k)\overline{u(k)} e^{-2\pi i k n/N}.$$

## Goal

Given a vector valued $N$-periodic code $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$, construct the following in a meaningful, computable way:

- Generalized $\mathbb{C}$-valued periodic ambiguity function $A_p^1(u) : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}$
- $\mathbb{C}^d$-valued periodic ambiguity function $A_p^d(u) : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}^d$

The STFT is the *guide* and the *theory of frames* is the technology to obtain the goal.

# Outline



Norbert Wiener Center
for Harmonic Analysis and Applications

# Multiplication problem

- Given $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$.
- If $d = 1$ and $e_n = e^{2\pi i n/N}$, then

$$A_p(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\langle u(m+k), u(k)e_{nk}\rangle.$$

## Multiplication problem

To characterize sequences $\{E_k\} \subseteq \mathbb{C}^d$ and multiplications $*$ so that

$$A_p^1(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\langle u(m+k), u(k) * E_{nk}\rangle \in \mathbb{C}$$

is a meaningful and well-defined *ambiguity function.* This formula is clearly motivated by the STFT.

# Ambiguity function assumptions

There is a natural way to address the multiplication problem motivated by the fact that $e_m e_n = e_{m+n}$. To this end, we shall make the *ambiguity function assumptions*:

- $\exists\ \{E_k\}_{k=0}^{N-1} \subseteq \mathbb{C}^d$ and a multiplication $*$ such that $E_m * E_n = E_{m+n}$ for $m, n \in \mathbb{Z}_N$;
- $\{E_k\}_{k=0}^{N-1} \subseteq \mathbb{C}^d$ is a tight frame for $\mathbb{C}^d$;
- $* : \mathbb{C}^d \times \mathbb{C}^d \longrightarrow \mathbb{C}^d$ is bilinear, in particular,

$$\left(\sum_{j=0}^{N-1} c_j E_j\right) * \left(\sum_{k=0}^{N-1} d_k E_k\right) = \sum_{j=0}^{N-1}\sum_{k=0}^{N-1} c_j d_k E_j * E_k.$$

# Calculation

- Let $\{E_j\}_j^{N-1} \subseteq \mathbb{C}^d$ satisfy the three ambiguity function assumptions.
- Given $u, v : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$ and $m, n \in \mathbb{Z}_N$.
- Then, one calculates

$$u(m) * v(n) \quad = \quad \frac{d^2}{N^2} \sum_{j=0}^{N-1} \sum_{s=0}^{N-1} \langle u(m), E_j \rangle \langle v(n), E_s \rangle E_{j+s}.$$

Norbert Wiener Center for Harmonic Analysis and Applications

# Outline



Norbert Wiener Center
for Harmonic Analysis and Applications

# $A_p^1(u)$ for DFT frames

- Let $\{E_j\}_j^{N-1} \subseteq \mathbb{C}^d$ satisfy the three ambiguity function assumptions.
- Further, assume that $\{E_j\}_{j=0}^{N-1}$ is a DFT frame, and let $r$ designate a fixed column.
- Without loss of generality, choose the first $d$ columns of the $N \times N$ DFT matrix.
- Then, one calculates

$$\begin{aligned} E_m * E_n(r) &= \frac{d^2}{N^2} \sum_{j=0}^{N-1} \sum_{s=0}^{N-1} \langle E_m, E_j \rangle \langle E_n, E_s \rangle E_{j+s}(r). \\ &= \frac{e_{(m+n)r}}{\sqrt{d}} = E_{m+n}(r). \end{aligned}$$

Norbert Wiener Center
for Harmonic Analysis and Applications

# $A^1_p(u)$ for DFT frames

- Thus, for DFT frames, $*$ is componentwise multiplication in $\mathbb{C}^d$ with a factor of $\sqrt{d}$.
- In this case $A^1_p(u)$ is well-defined for $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$ by

$$\begin{aligned} A^1_p(u)(m,n) &= \frac{1}{N}\sum_{k=0}^{N-1}\langle u(m+k), u(k) * E_{nk}\rangle \\ &= \frac{d}{N^2}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\langle E_j, u(k)\rangle\langle u(m+k), E_{j+nk}\rangle. \end{aligned}$$

Norbert Wiener Center for Harmonic Analysis and Applications

# Remark

- In the previous DFT example, $*$ is intrinsically related to the "addition" defined on the indices of the frame elements, viz., $E_m * E_n = E_{m+n}$.
- Alternatively, we could have $E_m * E_n = E_{m \bullet n}$ for some function $\bullet : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{Z}_N$, and, thereby, we could use frames which are not FUNTFs.
- Given a bilinear multiplication $* : \mathbb{C}^d \times \mathbb{C}^d \longrightarrow \mathbb{C}^d$, we can find a frame $\{E_j\}_j$ and an index operation $\bullet$ with the $E_m * E_n = E_{m \bullet n}$ property.
- If $\bullet$ is the multiplication for a group, possibly non-abelian and/or infinite, we may reverse the process and find a FUNTF and bilinear multiplication $*$ with the $E_m * E_n = E_{m \bullet n}$ property.

Norbert Wiener Center for Harmonic Analysis and Applications

# $A^1_p(u)$ for cross product frames

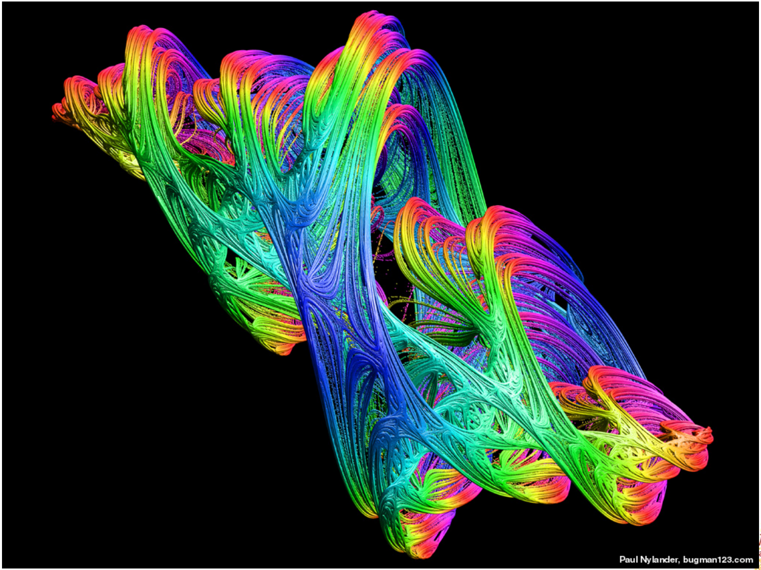
- Take $* : \mathbb{C}^3 \times \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ to be the cross product on $\mathbb{C}^3$ and let $\{i, j, k\}$ be the standard basis.
- $i * j = k$, $j * i = -k$, $k * i = j$, $i * k = -j$, $j * k = i$, $k * j = -i$, $i * i = j * j = k * k = 0$. $\{0, i, j, k, -i, -j, -k,\}$ is a tight frame for $\mathbb{C}^3$ with frame constant 2. Let $E_0 = 0, E_1 = i, E_2 = j, E_3 = k, E_4 = -i, E_5 = -j, E_6 = -k$.
- The index operation corresponding to the frame multiplication is the non-abelian operation $\bullet : \mathbb{Z}_7 \times \mathbb{Z}_7 \longrightarrow \mathbb{Z}_7$, where $1 \bullet 2 = 3, 2 \bullet 1 = 6, 3 \bullet 1 = 2, 1 \bullet 3 = 5, 2 \bullet 3 = 1, 3 \bullet 2 = 4, 1 \bullet 1 = 2 \bullet 2 = 3 \bullet 3 = 0, n \bullet 0 = 0 \bullet n = 0, 1 \bullet 4 = 0, 1 \bullet 5 = 6, 1 \bullet 6 = 2, 4 \bullet 1 = 0, 5 \bullet 1 = 3, 6 \bullet 1 = 5, 2 \bullet 4 = 3, 2 \bullet 5 = 0$, etc.
- The three ambiguity function assumptions are valid and so we can write the cross product as

$$u \times v = u * v = \frac{1}{2^2} \sum_{s=1}^{6} \sum_{t=1}^{6} \langle u, E_s \rangle \langle v, E_t \rangle E_{s \bullet t}.$$

- Consequently, $A^1_p(u)$ can be well-defined.

Norbert Wiener Center
for Harmonic Analysis and Applications

# Inverse 4D Quaternion Julia Set



# Vector-valued ambiguity function $A_p^d(u)$

- Let $\{E_j\}_j^{N-1} \subseteq \mathbb{C}^d$ satisfy the three ambiguity function assumptions.
- Given $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$.
- The following definition is clearly *motivated* by the STFT.

**Definition**

$A_p^d(u) : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}^d$ is defined by

$$A_p^d(u)(m,n) = \frac{1}{N} \sum_{k=0}^{N-1} u(m+k) * \overline{u(k)} * \overline{E_{nk}}.$$

# Outline



Norbert Wiener Center for Harmonic Analysis and Applications

# STFT formulation of $A_p(u)$

- The discrete periodic ambiguity function of $u : \mathbb{Z}_N \longrightarrow \mathbb{C}$ can be written as

$$A_p(u)(m,n) = \frac{1}{N} \sum_{k=0}^{N-1} \langle \tau_m u(k), F^{-1}(\tau_n \hat{u})(k) \rangle,$$

  where $\tau_( m)u(k) = u(m+k)$ is translation by $m$ and $F^{-1}(u)(k)) = \check{u}(k)$ is Fourier inversion.
- As such we see that $A_p(u)$ has the form of a STFT.
- We shall develop a vector-valued DFT theory to *verify* (not just *motivate*) that $A_p^d(u)$ is an STFT in the case $\{E_k\}_{k=0}^{N-1}$ is a DFT frame for $\mathbb{C}^d$.

Norbert Wiener Center for Harmonic Analysis and Applications

# DFT frames and the vector-valued DFT

## Definition

Given $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$, and let $\{E_k\}_{k=0}^{N-1}$ be a DFT frame for $\mathbb{C}^d$. The *vector-valued discrete Fourier transform* of $u$ is

$$\forall n \in \mathbb{Z}_N, \; F(u)(n) = \hat{u}(n) = \sum_{m=0}^{N-1} u(m) * E_{mn},$$

where $*$ is pointwise (coordinatewise) multiplication.

- The vector-valued DFT inversion formula is valid if $N$ is prime.
- Vector-valued DFT uncertainty principle inequalities are valid, similar to Tao-Candes in compressive sensing.

Norbert Wiener Center for Harmonic Analysis and Applications

# Vector-valued Fourier inversion theorem

- Inversion process for the vector-valued case is analogous to the 1-dimensional case.
- We must define a new multiplication in the frequency domain to avoid divisibility problems.
- Define the weighted multiplication $(*) : \mathbb{C}^d \times \mathbb{C}^d \longrightarrow \mathbb{C}^d$ by $u(*)v = u * v * \omega$ where $\omega = (\omega_1, \ldots, \omega_d)$ has the property that each $\omega_n = \frac{1}{\#\{m \in \mathbb{Z}_N : mn = 0\}}$.
- For the following theorem assume $d << N$ or $N$ prime.

**Theorem - Vector-valued Fourier inversion**

The vector valued Fourier transform $F$ is an isomorphism from $\ell^2(\mathbb{Z}_N)$ to $\ell^2(\mathbb{Z}_N, \omega)$ with inverse

$$\forall\ m \in \mathbb{Z}_N, \quad F^{-1}(m) = u(m) = \frac{d}{N} \sum_{n=0}^{N-1} \hat{u}(n) * E_{-mn} * \omega.$$

$N$ prime implies $F$ is unitary.

# $A_p^d(u)$ as an STFT

- Given $u, v : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$, and let $\{E_k\}_{k=0}^{N-1}$ be a DFT frame for $\mathbb{C}^d$.
- $u * \overline{v}$ denotes pointwise (coordinatewise) multiplication with a factor of $\sqrt{d}$.
- We compute

$$A_p^d(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}(\tau_m u(k)) * \overline{F^{-1}(\tau_n \hat{u})(k)}.$$

- Thus, $A_p^d(u)$ is compatible with point of view of defining a vector-valued ambiguity function in the context of the STFT.

Norbert Wiener Center for Harmonic Analysis and Applications

# Outline

# Epilogue

- If $(G, \bullet)$ is a finite group with representation $\rho : G \longrightarrow GL(\mathbb{C}^d)$, then there is a frame $\{E_n\}_{n \in G}$ and bilinear multiplication, $* : \mathbb{C}^d \times \mathbb{C}^d \longrightarrow \mathbb{C}^d$, such that $E_m * E_n = E_{m \bullet n}$. Thus, we can develop $A_p^d(u)$ theory in this setting.
- Analyze ambiguity function behavior for (phase-coded) vector-valued waveforms $v : \mathbb{R} \longrightarrow \mathbb{C}^d$, defined by $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$ as

$$v = \sum_{k=0}^{N-1} u(k) \mathbb{1}_{[kT,(k+1)T)},$$

in terms of $A_p^d(u)$. (See Figure)

Norbert Wiener Center for Harmonic Analysis and Applications

# Computation of $u : \mathbb{Z}_N \to \mathbb{C}^d$ from ambiguity

- CAZAC and waveform computation of $u : \mathbb{Z}_N \to \mathbb{C}^d$ from A(u): Let $A_u$ be the $N \times N$ matix, $(A(u)(m,n))$. Define the $N \times N$ matrix $U = (U_{i,j})$, where $U_{i,j} = \langle u(i+j), u(j)\rangle$. Then

$$U = A_u D_N, \quad \text{where} \quad D_N = DFT \text{ matrix.}$$

- Let $d = 1$. Note that $U_{k,0} = u(k)\overline{u(0)}$. Hence, if we know the values of the ambiguity function, and, thus, the ambiguity function matrix $A_u$, then the sequence $u$, which generates it, can be computed as long as $u(0) \neq 0$. In fact, if $u(0) = 1$ then $u(k) = (A_u D_N)(k, 0)$.
- Similar result for $A_V(u)$ using our vector-valued Fourier analysis.
- Now we can address the classical *radar ambiguity problem*: Find the structure of all $z : \mathbb{Z}_N \to \mathbb{C}^d$ for which $|A(u)| = |A(z)|$ on $X \subseteq \mathbb{Z}_N \times \mathbb{Z}_N$.

Norbert Wiener Center
for Harmonic Analysis and Applications

# Computation of $u : \mathbb{Z}_N \to \mathbb{C}^d$ from ambiguity

- CAZAC and waveform computation of $u : \mathbb{Z}_N \to \mathbb{C}^d$ from A(u): Let $A_u$ be the $N \times N$ matix, $(A(u)(m,n))$. Define the $N \times N$ matrix $U = (U_{i,j})$, where $U_{i,j} = \langle u(i+j), u(j) \rangle$. Then

$$U = A_u D_N, \quad \text{where} \quad D_N = DFT \text{ matrix.}$$

- Let $d = 1$. If $u(0) = 1$ then $u(k) = (A_u D_N)(k, 0)$.
- Similar result for $A_V(u)$ using our vector-valued Fourier analysis.
- We are addressing the classical *radar ambiguity problem*: Find the structure of all $z : \mathbb{Z}_N \to \mathbb{C}^d$ for which $|A(u)| = |A(z)|$ on $X \subseteq \mathbb{Z}_N \times \mathbb{Z}_N$. This is not even resolved for the narrow-band case.
- The radar ambiguity problem is closely related to our approach of achieving diverse ambiguity function behavior.

Thats all folks!

Norbert Wiener Center
for Harmonic Analysis and Applications

Norbert Wiener Center
for Harmonic Analysis and Applications