Waveform design and balayage

Dedicated to the memory of Mitch Taibleson

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Narrow band ambiguity functions and CAZAC codes

Discrete ambiguity functions

Let $u : \{0, 1, \ldots, N-1\} \to \mathbb{C}$.

- $u_p : \mathbb{Z}_N \to \mathbb{C}$ is the N-periodic extension of u.
- $u_a : \mathbb{Z} \to \mathbb{C}$ is an aperiodic extension of u:

$$u_a[m] = \begin{cases} u[m], & m = 0, 1, \ldots, N-1 \\ 0, & \text{otherwise.} \end{cases}$$

- The *discrete periodic ambiguity function* $A_p(u) : \mathbb{Z}_N \times \mathbb{Z}_N \to \mathbb{C}$ of u is

$$A_p(u)(m,n) = \frac{1}{N} \sum_{k=0}^{N-1} u_p[m+k]\overline{u_p[k]}e^{2\pi ikn/N}.$$

- The *discrete aperiodic ambiguity* function $A_a(u) : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ of u is

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The ambiguity function

- The *complex envelope* w of the *phase coded waveform* $Re(w)$ associated to a unimodular N-periodic sequence $u : \mathbb{Z}_N \to \mathbb{C}$ is

$$w(t) = \frac{1}{\sqrt{\tau}} \sum_{k=0}^{N-1} u[k] \, \mathbb{1}\left(\frac{t - kt_b}{t_b}\right),$$

where $\mathbb{1}$ is the characteristic function of the interval $[0, 1)$, τ is the pulse duration, and $t_b = \tau / N$.

- For spectral shaping problems, smooth replacements to $\mathbb{1}$ are analyzed.
- The *(aperiodic) ambiguity function* $\mathcal{A}(w)$ of w is

$$\mathcal{A}(w)(t, \gamma) = \int w(s + t)\overline{w(s)} e^{2\pi i s\gamma} ds,$$

where $t \in \mathbb{R}$ is time delay and $\gamma \in \widehat{\mathbb{R}}(= \mathbb{R})$ is frequency shift.

CAZAC sequences

- $u : \mathbb{Z}_N \to \mathbb{C}$ is *Constant Amplitude Zero Autocorrelation (CAZAC)*:

$$\forall m \in \mathbb{Z}_N, \quad |u[m]| = 1, \text{ (CA)}$$

  and

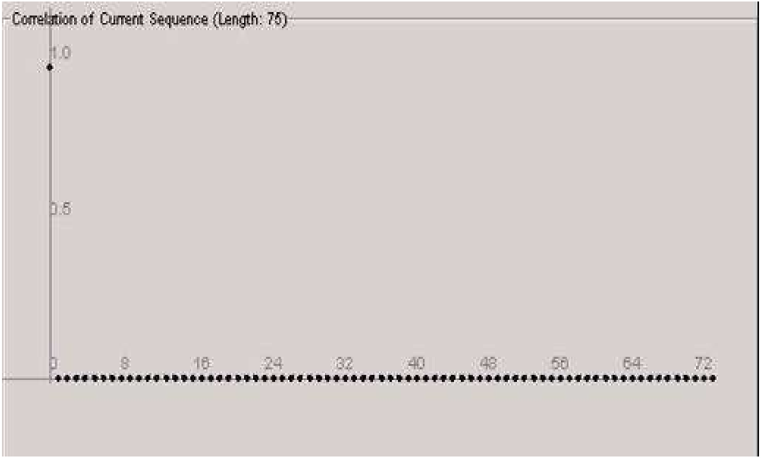
$$\forall m \in \mathbb{Z}_N \setminus \{0\}, \quad A_p(u)(m, 0) = 0. \text{ (ZAC)}$$

- Empirically, the (ZAC) property of CAZAC sequences u leads to phase coded waveforms w with low *aperiodic autocorrelation* $\mathcal{A}(w)(t, 0)$.
- Are there only finitely many non-equivalent CAZAC sequences?
  - "Yes" for N prime and "No" for $N = MK^2$,
  - Generally unknown for N square free and not prime.

Examples of CAZAC codes

$$K = 75 : u(x) =$$
$$(1, 1, 1, 1, 1, 1, e^{2\pi i\frac{1}{15}}, e^{2\pi i\frac{2}{15}}, e^{2\pi i\frac{1}{5}}, e^{2\pi i\frac{4}{15}}, e^{2\pi i\frac{1}{3}}, e^{2\pi i\frac{7}{15}}, e^{2\pi i\frac{3}{5}},$$
$$e^{2\pi i\frac{11}{15}}, e^{2\pi i\frac{13}{15}}, 1, e^{2\pi i\frac{1}{5}}, e^{2\pi i\frac{2}{5}}, e^{2\pi i\frac{3}{5}}, e^{2\pi i\frac{4}{5}}, 1, e^{2\pi i\frac{4}{15}}, e^{2\pi i\frac{8}{15}}, e^{2\pi i\frac{4}{5}},$$
$$e^{2\pi i\frac{16}{15}}, e^{2\pi i\frac{1}{3}}, e^{2\pi i\frac{2}{3}}, e^{2\pi i}, e^{2\pi i\frac{4}{3}}, e^{2\pi i\frac{5}{3}}, 1, e^{2\pi i\frac{2}{5}}, e^{2\pi i\frac{4}{5}}, e^{2\pi i\frac{6}{5}},$$
$$e^{2\pi i\frac{8}{5}}, 1, e^{2\pi i\frac{7}{15}}, e^{2\pi i\frac{14}{15}}, e^{2\pi i\frac{7}{5}}, e^{2\pi i\frac{28}{15}}, e^{2\pi i\frac{1}{3}}, e^{2\pi i\frac{13}{15}}, e^{2\pi i\frac{7}{5}}, e^{2\pi i\frac{29}{15}},$$
$$e^{2\pi i\frac{37}{15}}, 1, e^{2\pi i\frac{3}{5}}, e^{2\pi i\frac{6}{5}}, e^{2\pi i\frac{9}{5}}, e^{2\pi i\frac{12}{5}}, 1, e^{2\pi i\frac{2}{3}}, e^{2\pi i\frac{4}{3}}, e^{2\pi i\cdot 2}, e^{2\pi i\frac{8}{3}},$$
$$e^{2\pi i\frac{1}{3}}, e^{2\pi i\frac{16}{15}}, e^{2\pi i\frac{9}{5}}, e^{2\pi i\frac{38}{15}}, e^{2\pi i\frac{49}{15}}, 1, e^{2\pi i\frac{4}{5}}, e^{2\pi i\frac{8}{5}}, e^{2\pi i\frac{12}{5}}, e^{2\pi i\frac{16}{5}},$$
$$1, e^{2\pi i\frac{13}{15}}, e^{2\pi i\frac{26}{15}}, e^{2\pi i\frac{13}{5}}, e^{2\pi i\frac{52}{15}}, e^{2\pi i\frac{1}{3}}, e^{2\pi i\frac{19}{15}}, e^{2\pi i\frac{11}{5}}, e^{2\pi i\frac{47}{15}}, e^{2\pi i\frac{61}{15}})$$

Autocorrelation of CAZAC $K = 75$



Wiener CAZAC codes

Rationale and theorem

Definition

$e^{\frac{2\pi ik}{M}} = \omega$ is a primitive *Mth root of unity* if $(k, M) = 1$.

Theorem 1

Given $N \geq 1$. Let

$$M = \begin{cases} N, & \text{N odd,} \\ 2N, & \text{N even,} \end{cases}$$

and let ω be a primitive Mth root of unity. Define the Wiener sequence $u : \mathbb{Z}_N \to \mathbb{C}$ by $u(k) = \omega^{k^2}, 0 \leq k \leq N-1$. Then u is a CAZAC sequence.

Rationale and theorem

Corollary

Let $\{u(k)\}_{k=0}^{N-1}$ be a Wiener CAZAC waveform as given in Theorem 1. (In particular, ω is a primitive M-th root of unity.)
If N is even, then

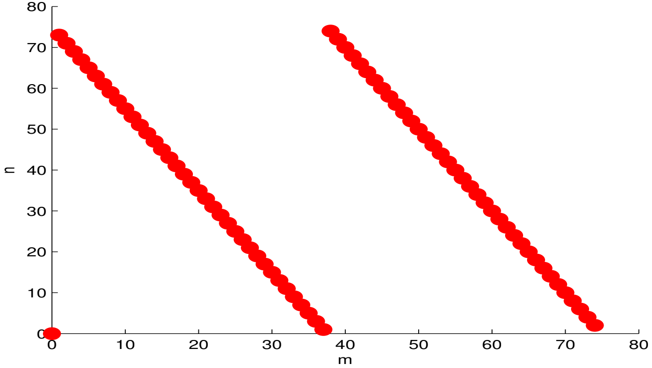
$$A_u(m,n) = \begin{cases} \omega^{m^2}, & m \equiv -n \; mod \; N, \\ 0, & \text{otherwise.} \end{cases}$$

If N is odd, then

$$A_u(m,n) = \begin{cases} \omega^{m^2}, & m \equiv -n(N+1)/2 \; mod \; N, \\ 0, & \text{otherwise.} \end{cases}$$

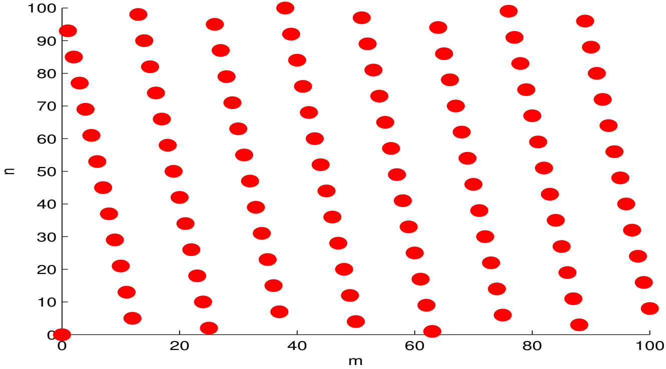
Wiener CAZAC ambiguity domain

$K = 75, j = 1$



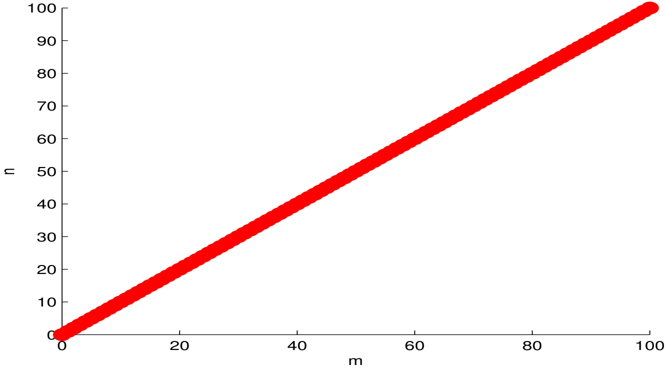
Wiener CAZAC ambiguity domain

$K = 101, j = 4$



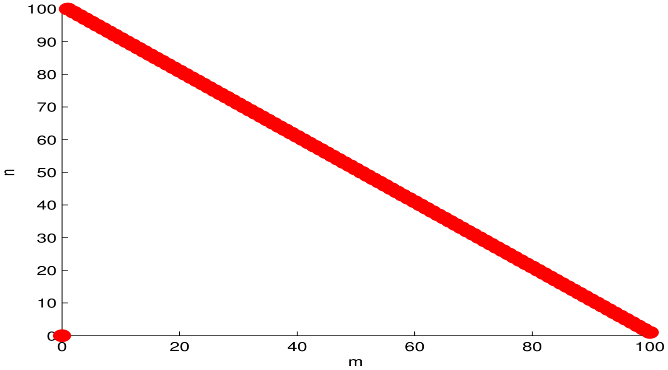
Wiener CAZAC ambiguity domain

$K = 101, j = 50$



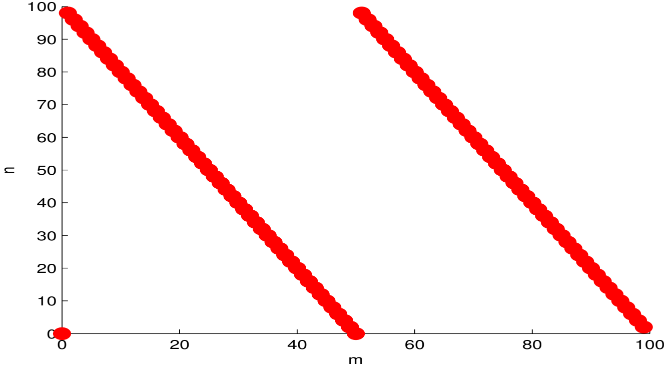
Wiener CAZAC ambiguity domain

$K = 101, j = 51$



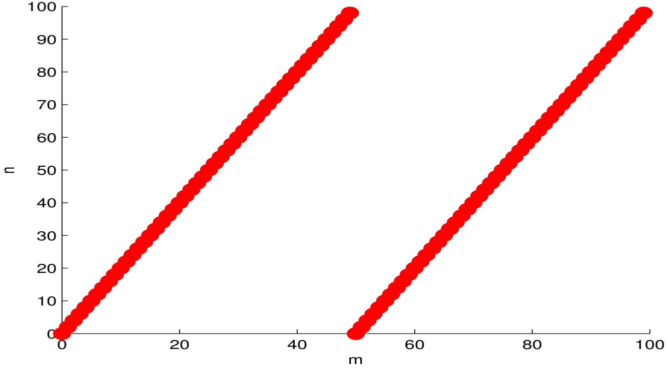
Wiener CAZAC ambiguity domain

$K = 100, j = 2$



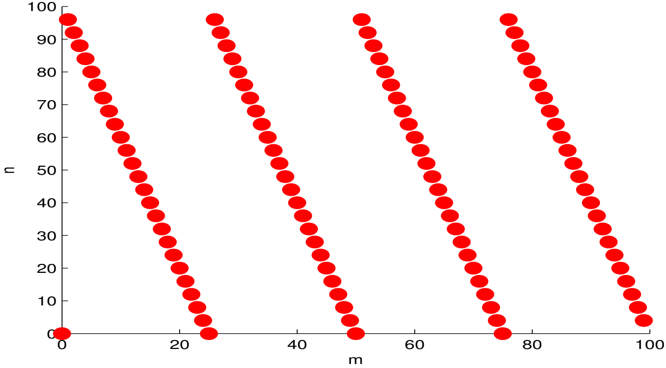
Wiener CAZAC ambiguity domain

$K = 100, j = 98$



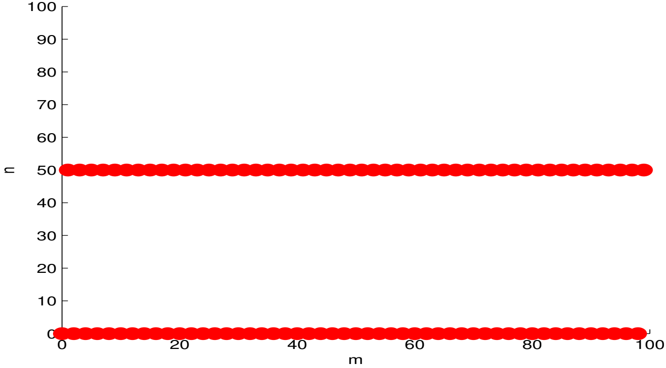
Wiener CAZAC ambiguity domain

$K = 100, j = 4$



Wiener CAZAC ambiguity domain

$K = 100, j = 50$



Björck CAZAC codes and ambiguity function comparisons

Legendre symbol

Let N be a prime and $(k, N) = 1$.

- k is a quadratic residue mod N if $x^2 = k \pmod{N}$ has a solution.
- k is a quadratic non–residue mod N if $x^2 = k \pmod{N}$ has no solution.
- The Legendre symbol:

$$\left(\frac{k}{N}\right) = \begin{cases} 1, & \text{if } k \text{ is a quadratic residue mod } N\,, \\ -1, & \text{if } k \text{ is a quadratic non–residue mod } N. \end{cases}$$

The diagonal of the product table of $\mathbb{Z}_N$ gives values $k \in \mathbb{Z}$ which are squares. As such we can program Legendre symbol computation.

Example: $N = 7$. $(\frac{k}{N}) = 1$ if $k = 1, 2, 4$.

Definition

Let N be a prime number. A *Björck CAZAC sequence* of length N is

$$u[k] = e^{i\theta_N(k)}, \quad k = 0, 1, \ldots, N-1,$$

where, for $N = 1 \pmod 4$,

$$\theta_N(k) = \arccos\left(\frac{1}{1+\sqrt{N}}\right)\left(\frac{k}{N}\right),$$

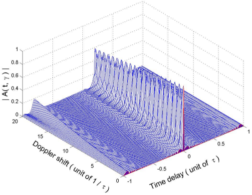
and, for $N = 3 \pmod 4$,

$$\theta_N(k) = \frac{1}{2}\arccos\left(\frac{1-N}{1+N}\right)\left[\left(1-\delta_k\right)\left(\frac{k}{N}\right) + \delta_k\right].$$

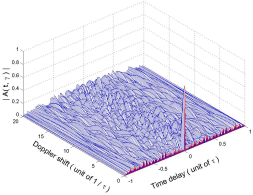
δ_k is Kronecker delta and $\left(\frac{k}{N}\right)$ is Legendre symbol.

Quadratic and Björck ambiguity comparison

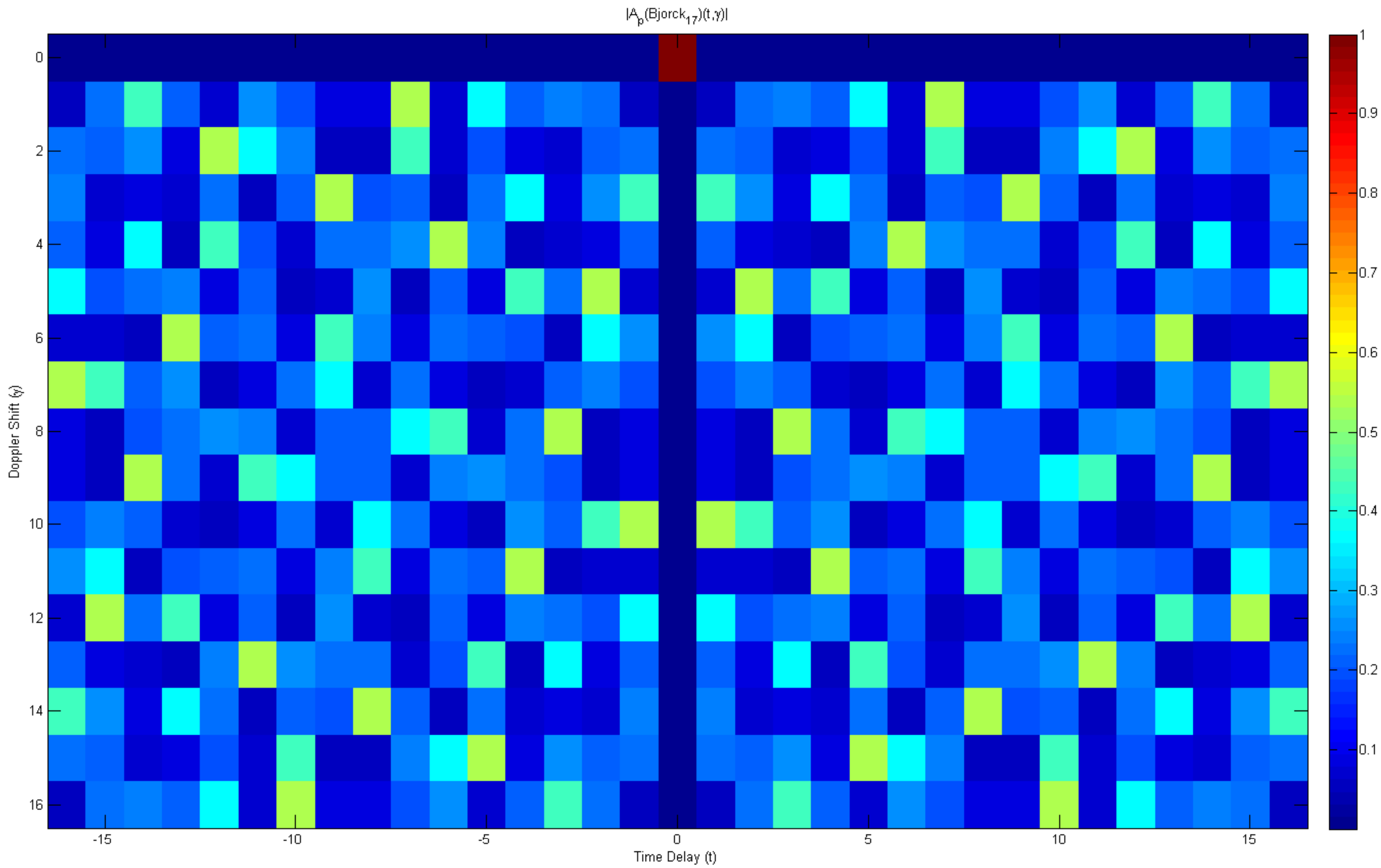
- Waveforms associated to Chu-Zadoff and P4 CAZACs are known for their low sidelobes at zero Doppler shift, but their ambiguity functions exhibit strong coupling in the time-frequency plane.
- Waveforms associated to Björck CAZACs can more effectively decouple the effect of time and frequency shifts. However, at zero Doppler shift, their sidelobe behavior is less desirable than quadratic phase CAZACs.
- These differences led to our concatenation idea.



Chu-Zadoff 101



Björck 101



Absolute value of Bjorck code of length 17

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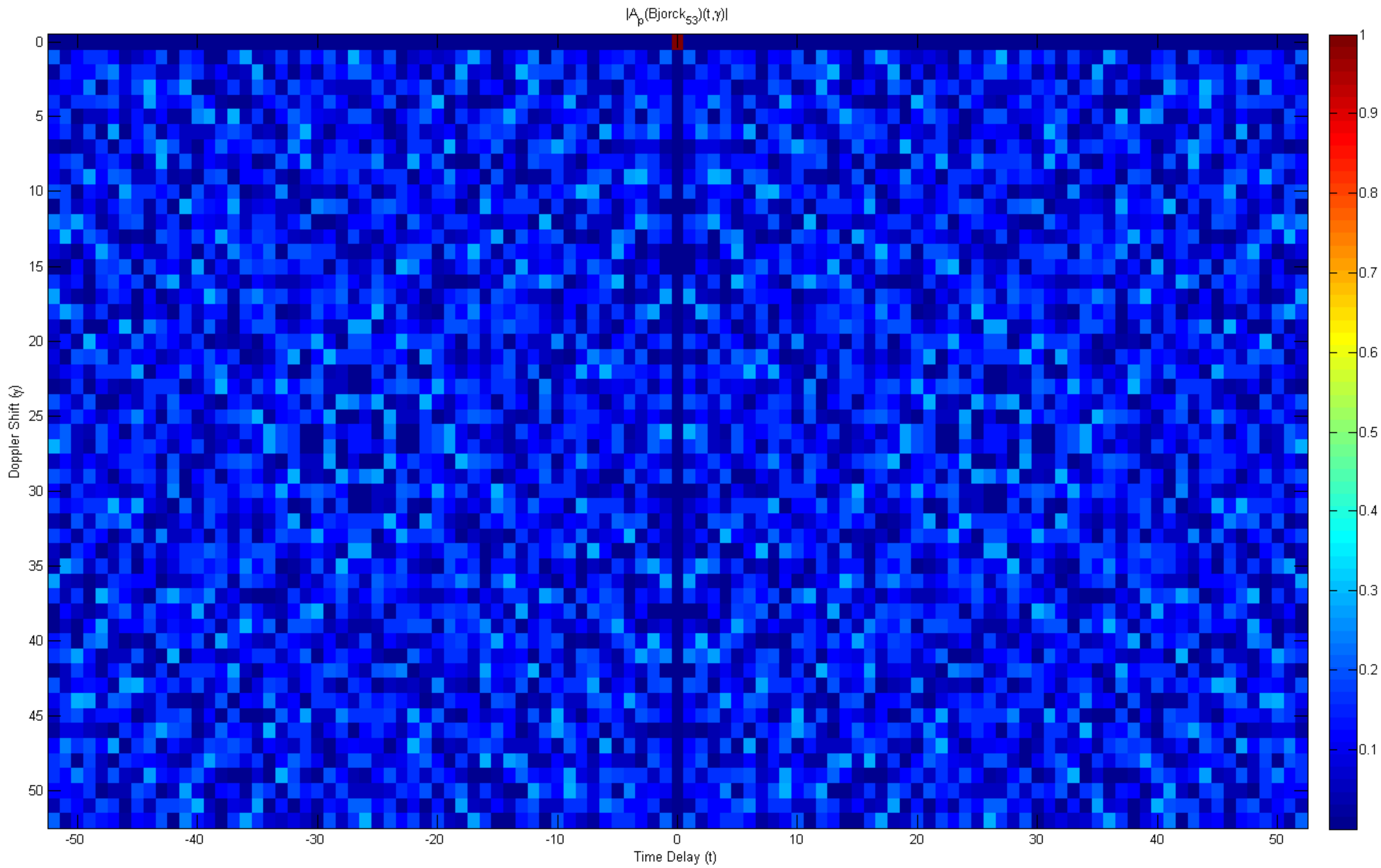
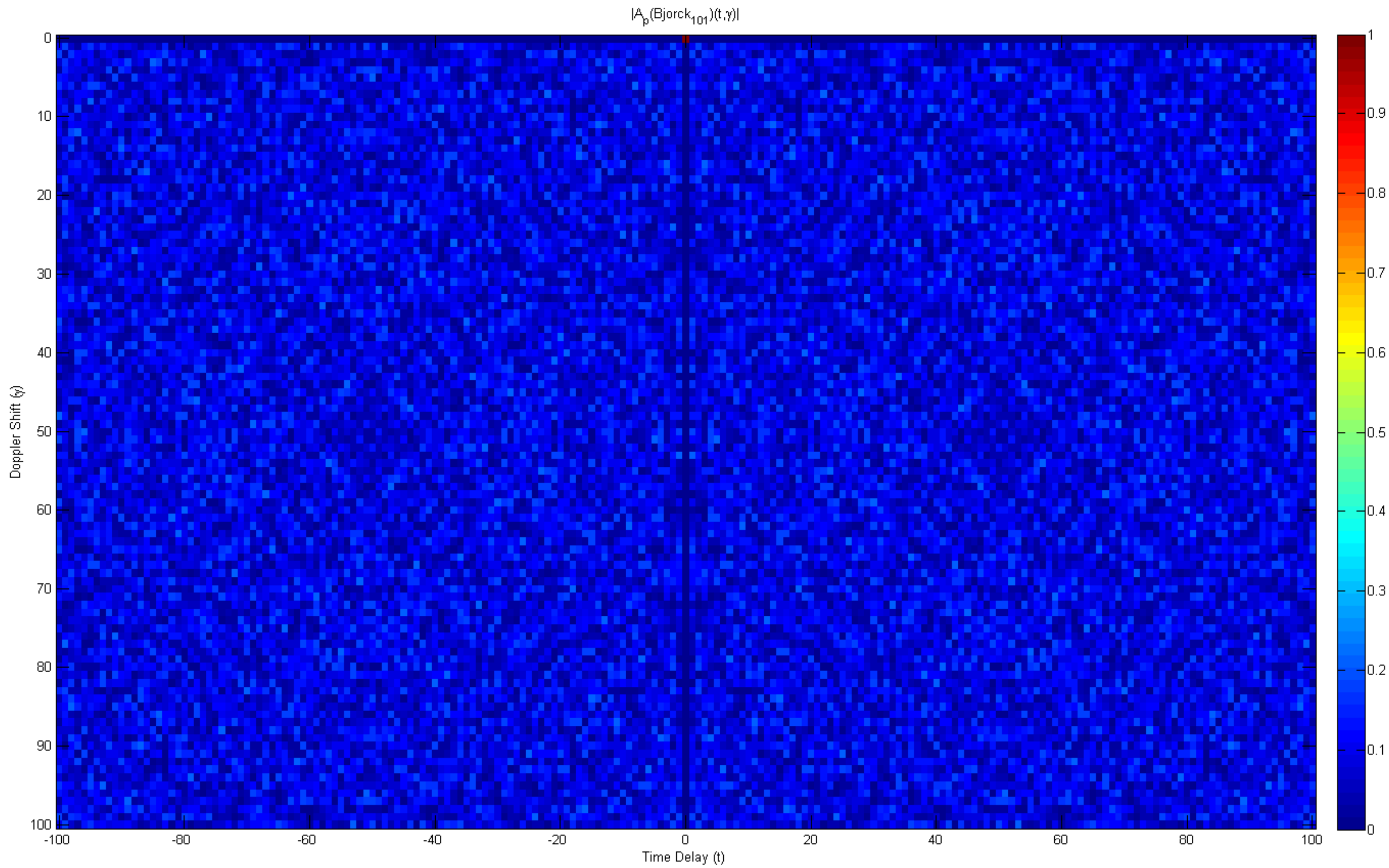


Absolute value of Bjorck code of length 53



Absolute value of Bjorck code of length 101

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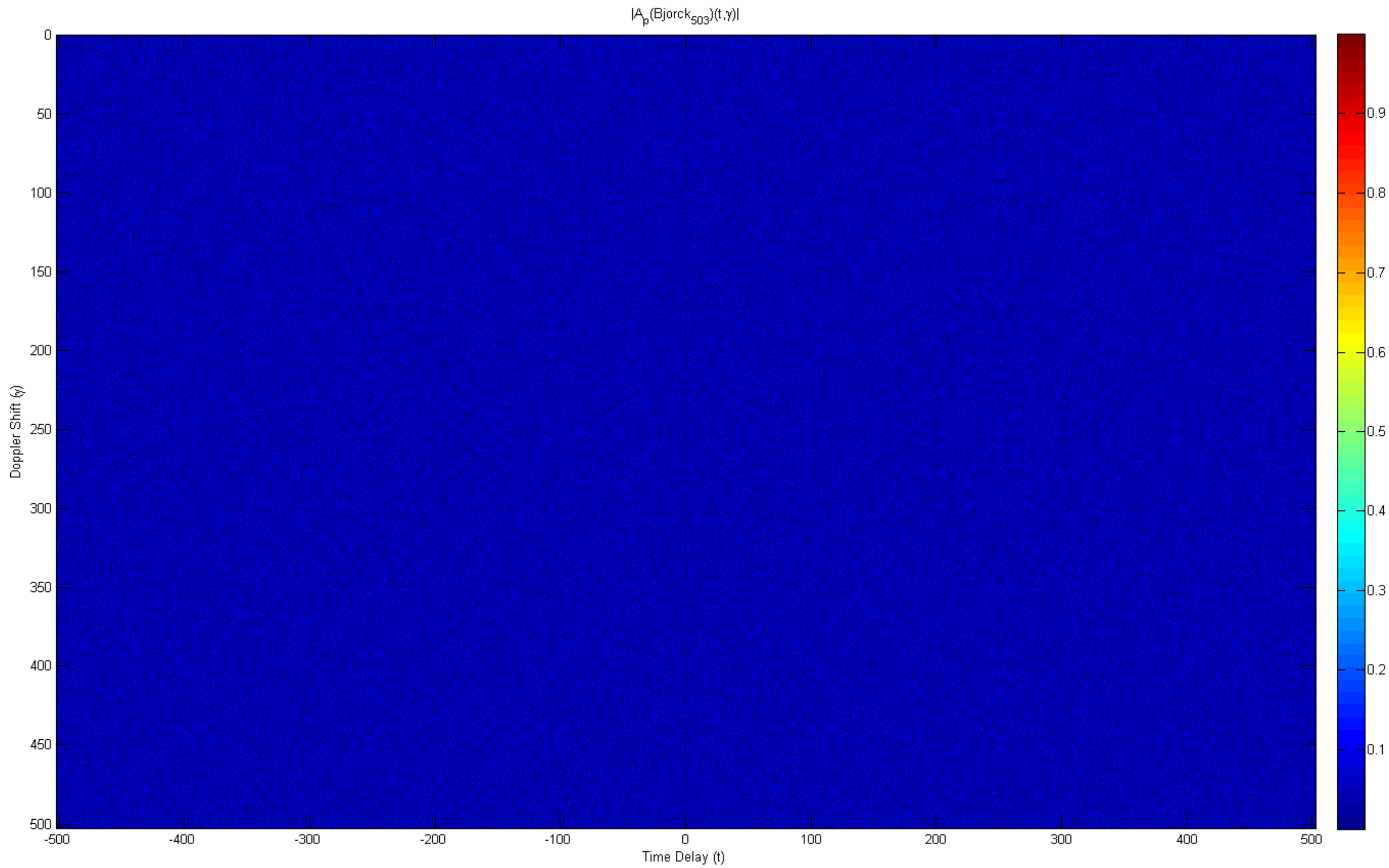

Absolute value of Bjorck code of length 503

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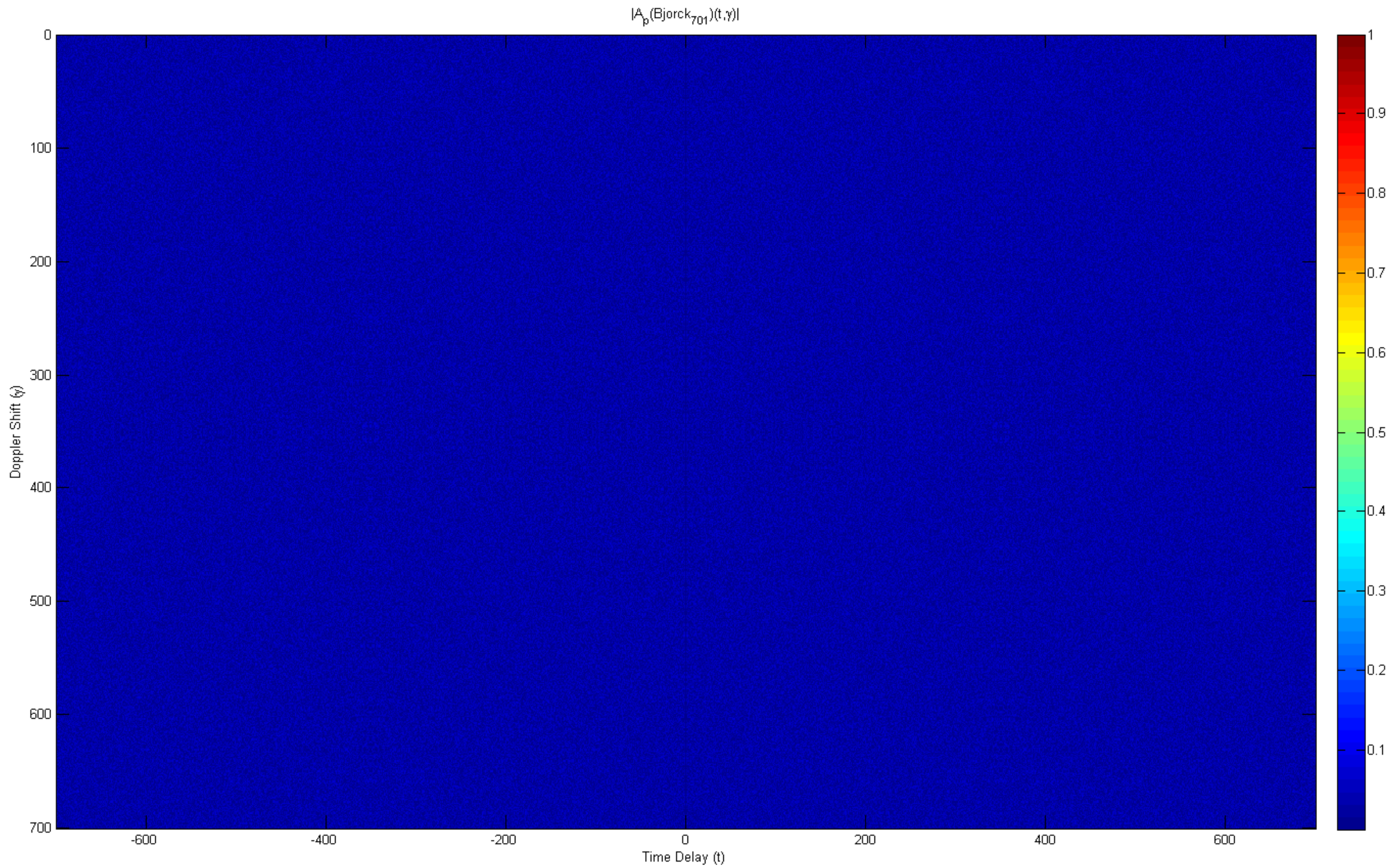
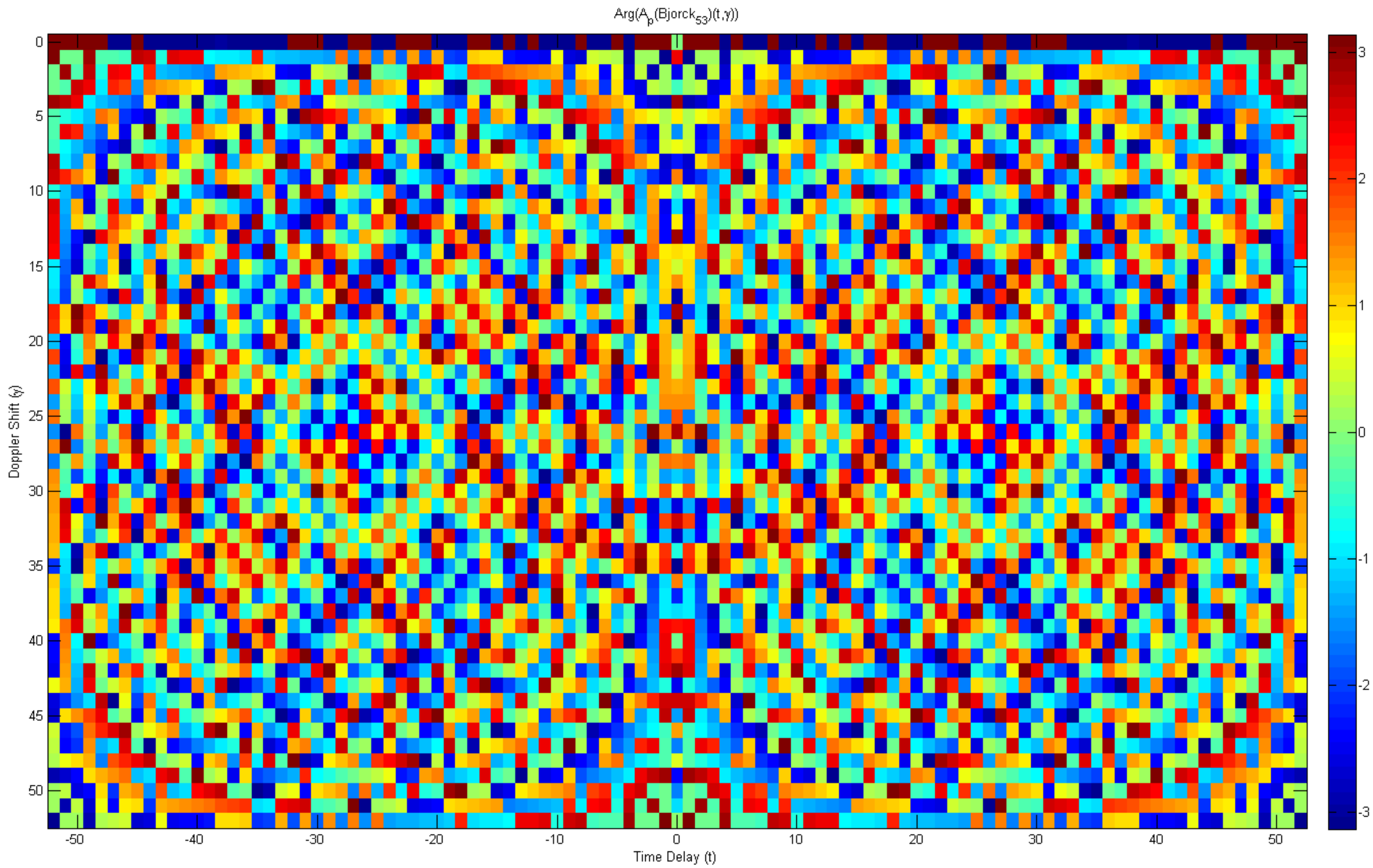


Absolute value of Bjorck code of length 701



Argument of Bjorck code of length 53

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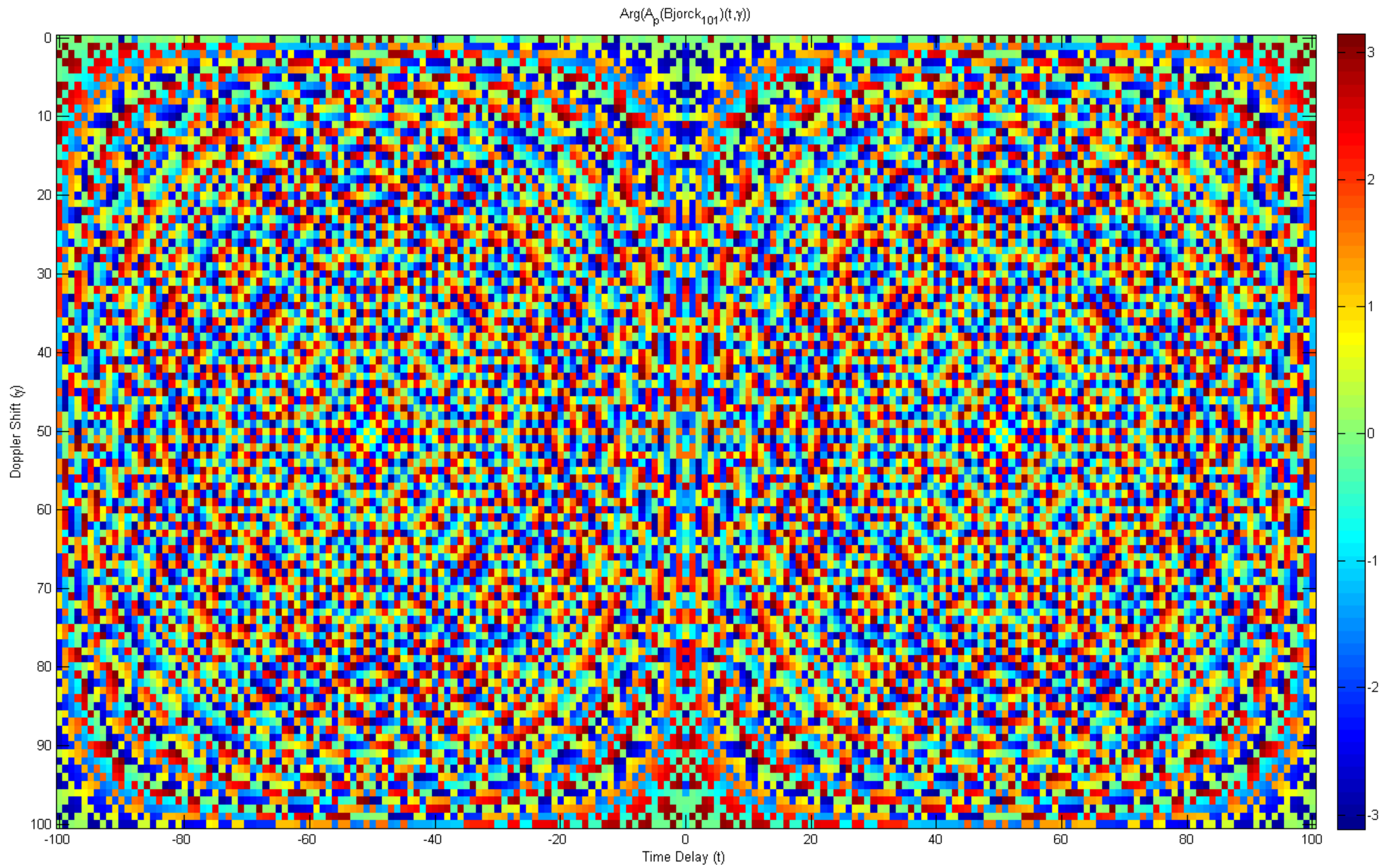

Argument of Bjorck code of length 101

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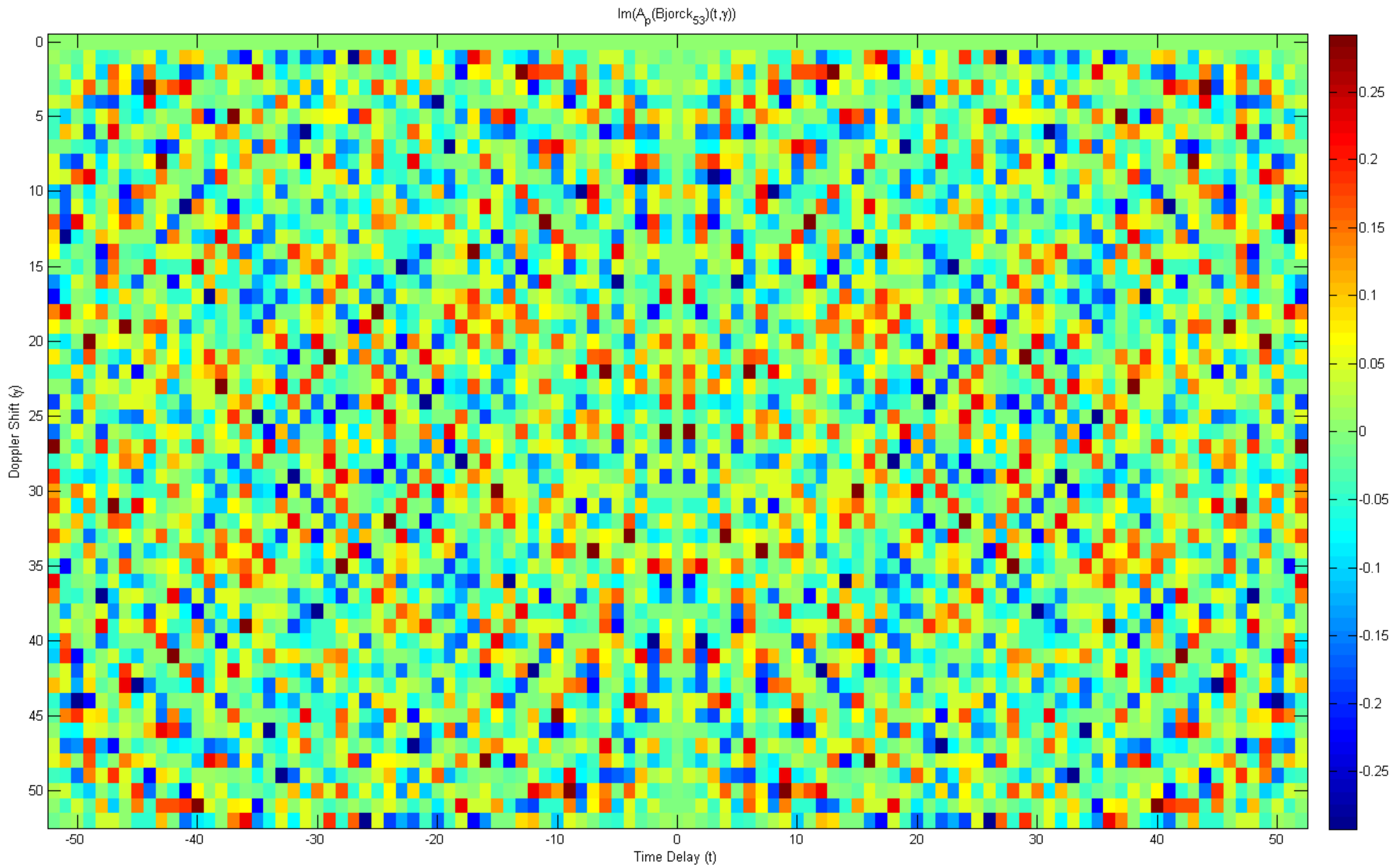
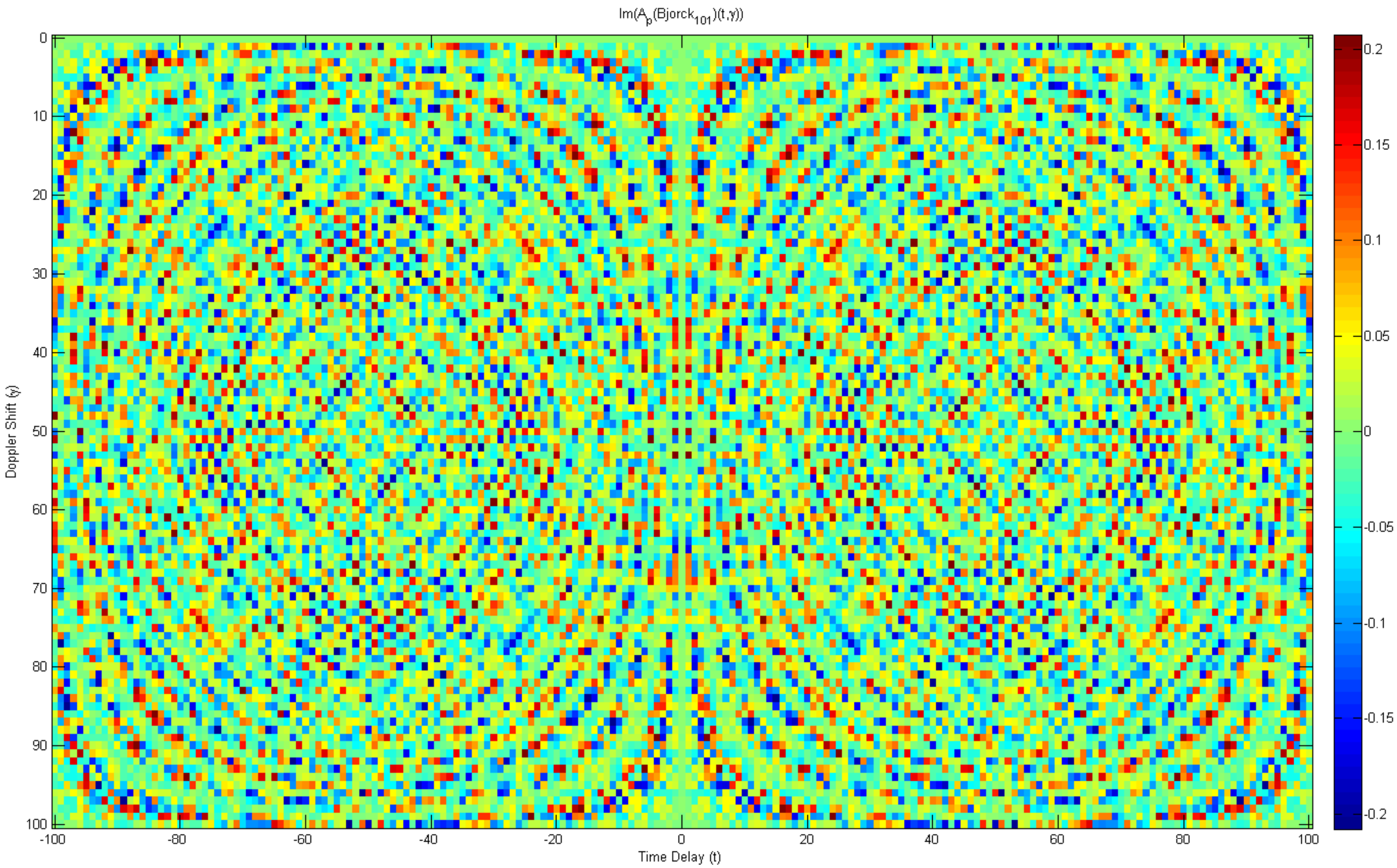


Imaginary Part of Bjorck code of length 53



Imaginary Part of Bjorck code of length 101

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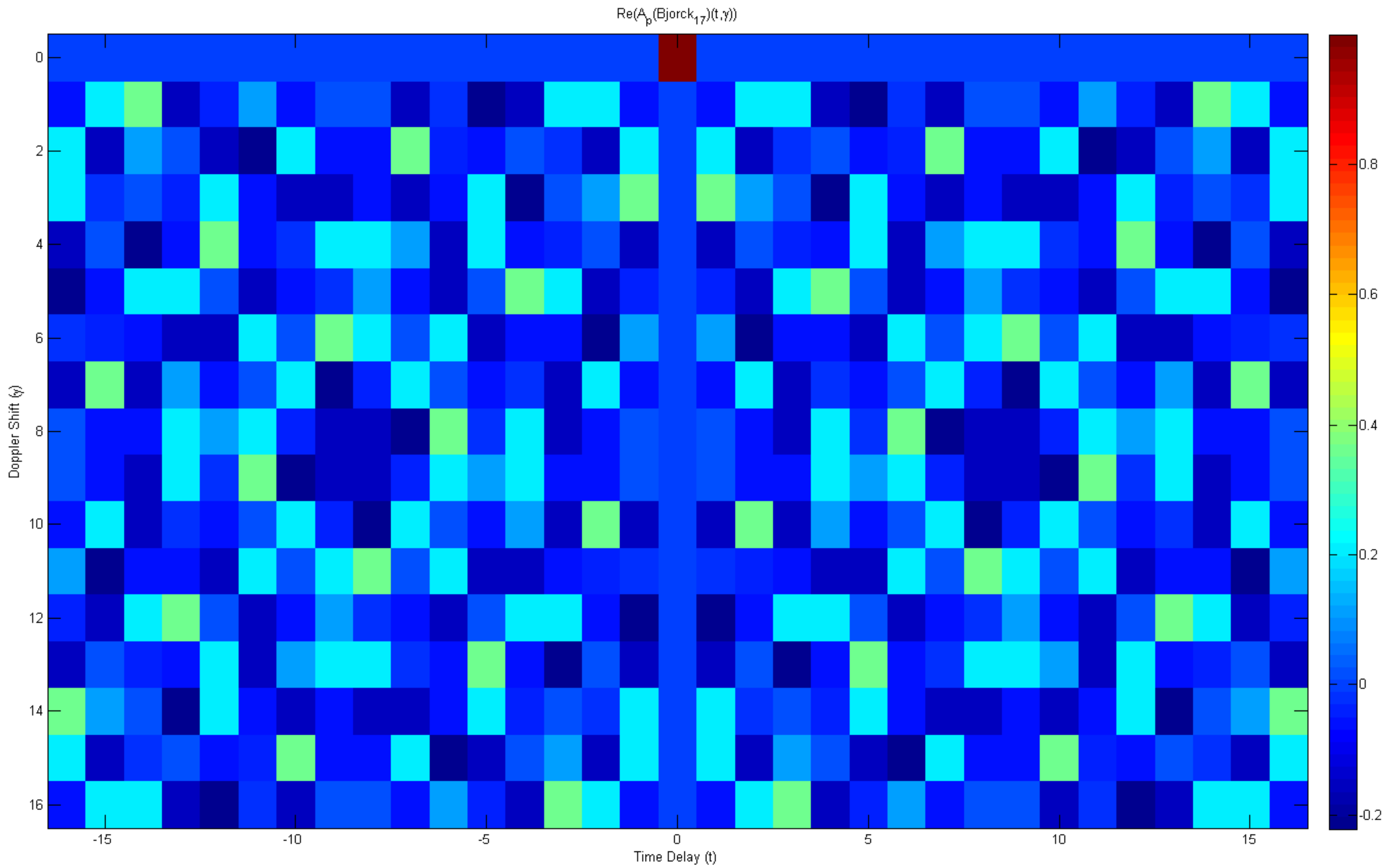

Real Part of Bjorck code of length 17

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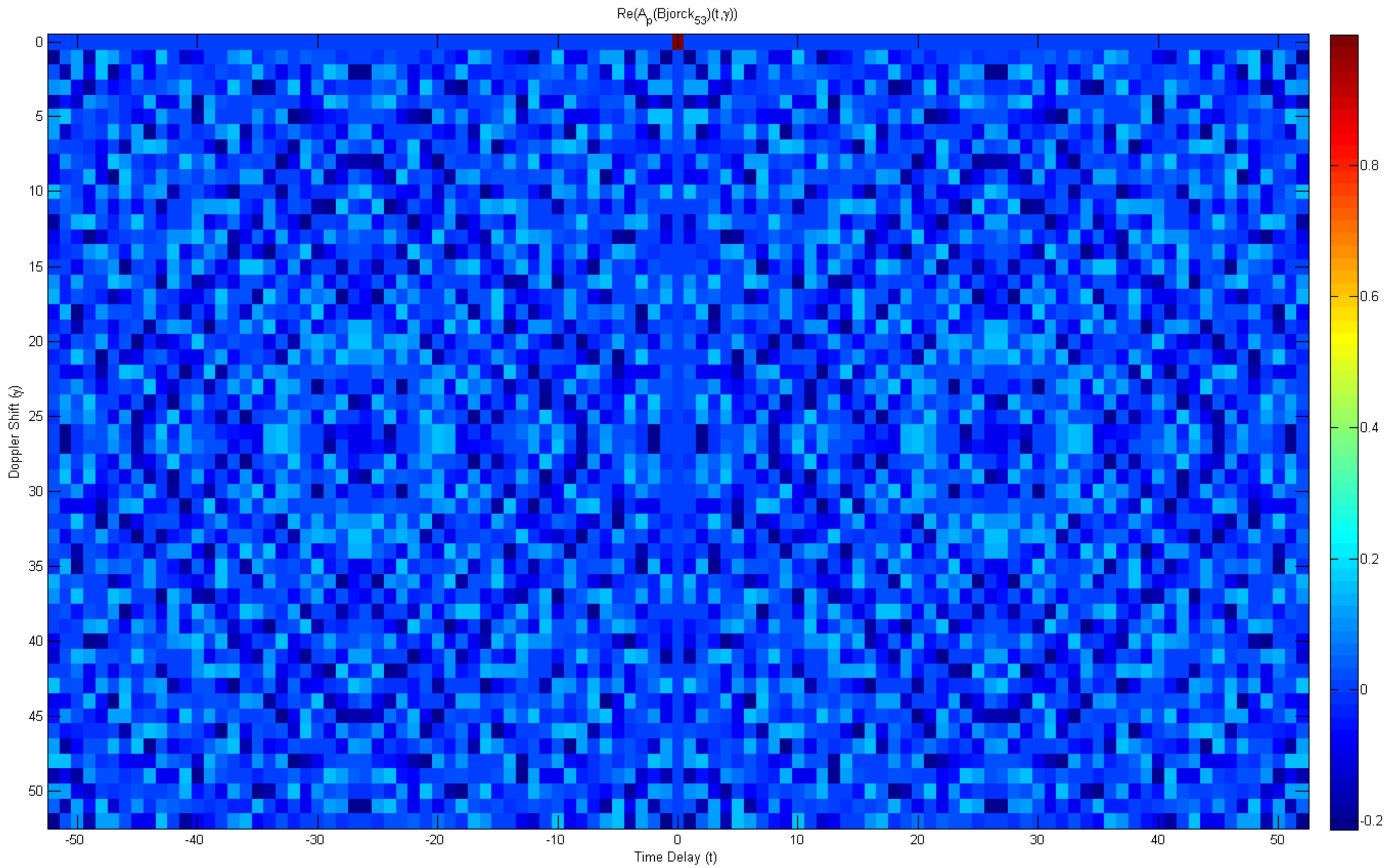

Real Part of Bjorck code of length 53

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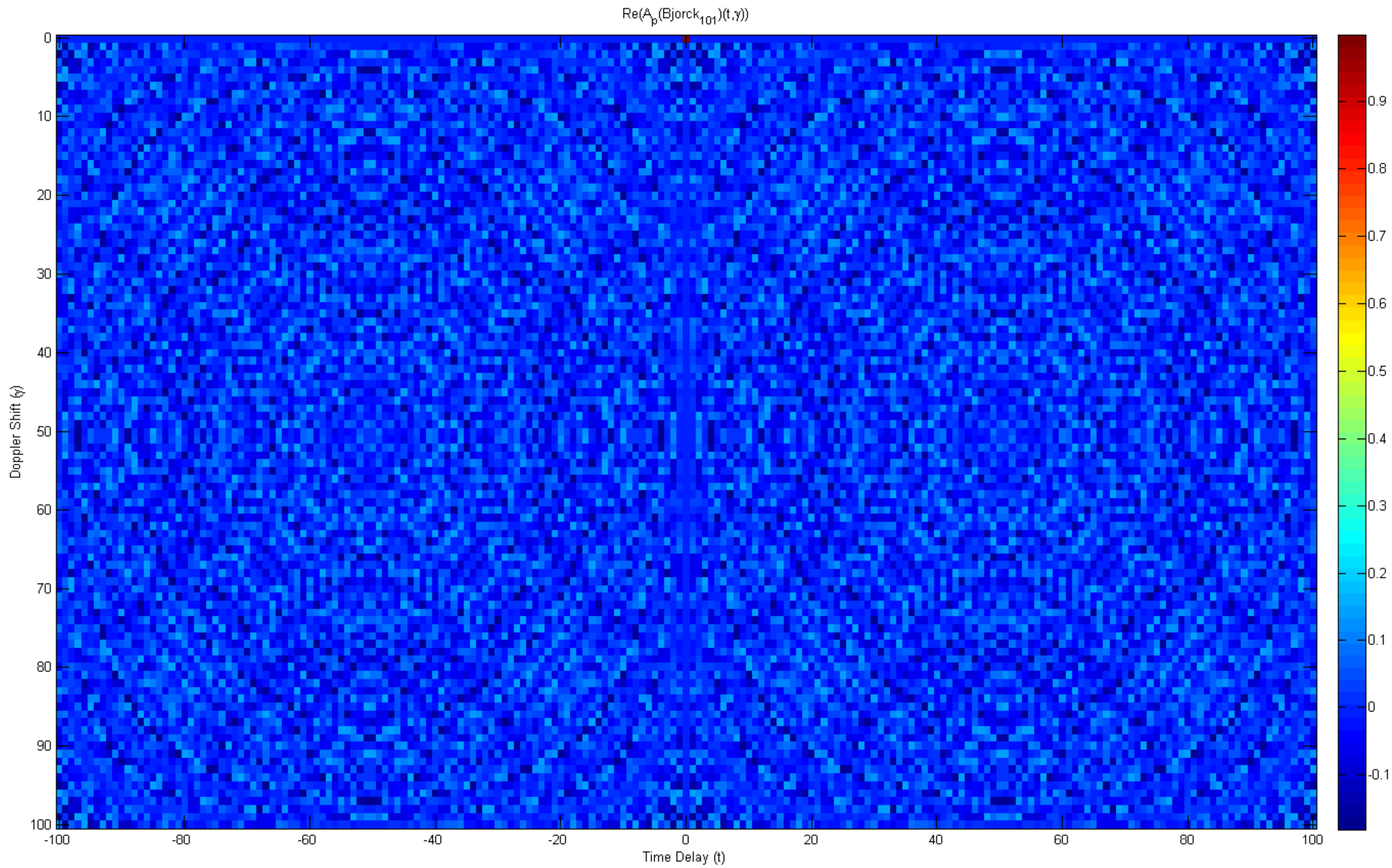

Real Part of Bjorck code of length 101

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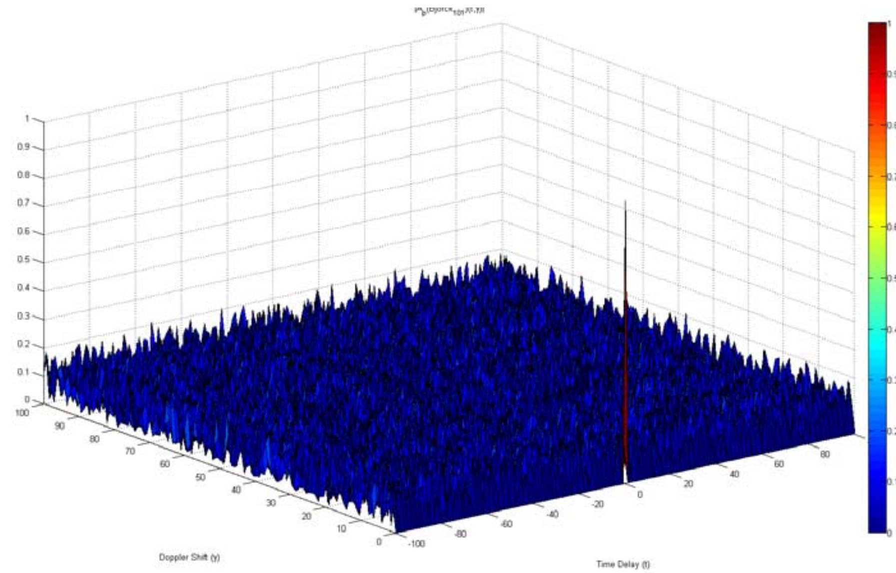
Björck CAZAC Discrete Narrow-band Ambiguity Function

Let u_p denote the Björck CAZAC sequence for prime p, and let $A(u_p)$ be its discrete narrow-band ambiguity function defined on $\mathbb{Z}_N \times \mathbb{Z}_N$.

<u>**Theorem**</u> (Benedetto-Woodworth) $\forall \epsilon > 0, \exists p_\epsilon$, a prime number which can be calculated, such that $\forall p \geq p_\epsilon$, p prime, and $\forall (m,n) \in \mathbb{Z}_N \times \mathbb{Z}_N \backslash (0,0)$, $|A(u_p)(m,n)| < \epsilon$.



Perspective

Sequences for coding theory, cryptography, phase-coded waveforms, and communications (synchronization, fast start-up equalization, frequency hopping) include the following in the periodic case:

- Gauss, Wiener (1927), Zadoff (1963), Schroeder (1969), Chu (1972), Zhang and Golomb (1993)
- Frank (1953), Zadoff and Abourezk (1961), Heimiller (1961)
- Milewski (1983)
- Björck (1985) and Golomb (1992),

and their generalizations, both periodic and aperiodic.

The general problem of using codes to generate signals leads to frames.

Frames

FUNTF

- A set $F = \{e_j\}_{j \in J} \subseteq \mathbb{F}^d$ is a *frame* for $\mathbb{F}^d$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, if

$$\exists\, A, B > 0 \quad \text{such that} \quad \forall\, x \in \mathbb{F}^d, \quad A\|x\|^2 \leq \sum_{j \in J} |\langle x, e_j \rangle|^2 \leq B\|x\|^2.$$

- F *tight* if $A = B$. A finite unit-norm tight frame F is a FUNTF.
- N row vectors from any fixed $N \times d$ submatrix of the $N \times N$ DFT matrix, $\frac{1}{\sqrt{d}}(e^{2\pi imn/N})$, is a FUNTF for $\mathbb{C}^d$.
- If F is a FUNTF for $\mathbb{F}^d$, then

$$\forall x \in \mathbb{F}^d, \quad x = \frac{d}{N} \sum_{j=1}^{N} \langle x, e_j \rangle e_j.$$

- Frames: redundant representation, compensate for hardware errors, inexpensive, numerical stability, minimize effects of noise.

DFT FUNTFs

- $N \times d$ submatrices of the $N \times N$ DFT matrix are FUNTFs for $\mathbb{C}^d$. These play a major role in finite frame $\Sigma\Delta$-quantization.

$$N = 8,\, d = 5 \qquad \frac{1}{\sqrt{5}}\begin{bmatrix} * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \end{bmatrix}$$

$$x_m = \frac{1}{5}(e^{2\pi i \frac{m}{8}}, e^{2\pi i m \frac{2}{8}}, e^{2\pi i m \frac{5}{8}}, e^{2\pi i m \frac{6}{8}}, e^{2\pi i m \frac{7}{8}})$$

$$m = 1, \ldots, 8.$$

- Sigma-Delta Super Audio CDs - but not all authorities are fans.

Properties and examples of FUNTFs

- Frames give redundant signal representation to compensate for hardware errors, to ensure numerical stability, and to minimize the effects of noise.
- Thus, if certain types of noises are known to exist, then the FUNTFs are constructed using this information.
- Orthonormal bases, vertices of Platonic solids, kissing numbers (sphere packing and error correcting codes) are FUNTFs.
- The vector-valued CAZAC – FUNTF problem: Characterize $u : \mathbb{Z}_K \longrightarrow \mathbb{C}^d$ which are CAZAC FUNTFs.

CAZACs and FUNTFs

- Let $u = \{u[k]\}_{k=1}^N$ be a CAZAC sequence in $\mathbb{C}$. Define

$$\forall\, k = 1, ..., N, \quad v_k = v[k] = \frac{1}{\sqrt{d}}(u[k], u[k+1], ..., u[k+d-1]).$$

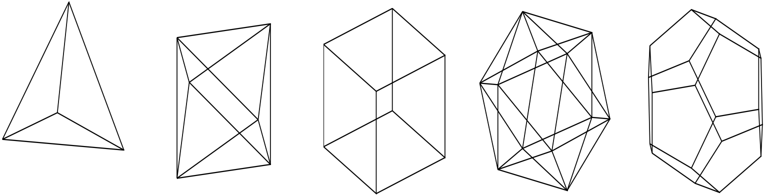
  Then $v = \{v[k]\}_{k=1}^N \subseteq \mathbb{C}^d$ is a CAZAC sequence in $\mathbb{C}^d$ and $\{v_k\}_{k=1}^N$ is a FUNTF for $\mathbb{C}^d$ with frame constant N/d.

- Let $\{x_k\}_{k=1}^N \subseteq \mathbb{C}^d$ be a FUNTF for $\mathbb{C}^d$, with frame constant A and with associated Bessel map $L : \mathbb{C}^d \to \ell^2(\mathbb{Z}_N)$; and let $u = \{u[j]\}_{j=1}^M \subseteq \mathbb{C}^d$ be a CAZAC sequence in $\mathbb{C}^d$. Then $\{\frac{1}{\sqrt{A}}L(u[j])\}_{j=1}^M \subseteq \mathbb{C}^N(= \ell^2(\mathbb{Z}_N)$ is a CAZAC sequence in $\mathbb{C}^N$.

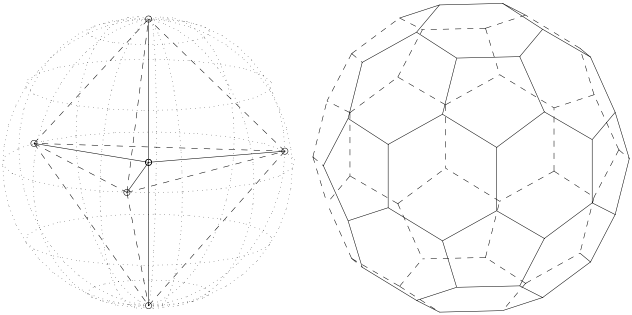
Recent applications of FUNTFs

- Robust transmission of data over erasure channels such as the internet [Casazza, Goyal, Kelner, Kovačević]
- Multiple antenna code design for wireless communications [Hochwald, Marzetta,T. Richardson, Sweldens, Urbanke]
- Multiple description coding [Goyal, Heath, Kovačević, Strohmer,Vetterli]
- Quantum detection [Bölcskei, Eldar, Forney, Oppenheim, Kebo, B]
- Grassmannian ”min-max” waveforms [Calderbank, Conway, Sloane, et al., Kolesar, B]

Recent applications of FUNTFs



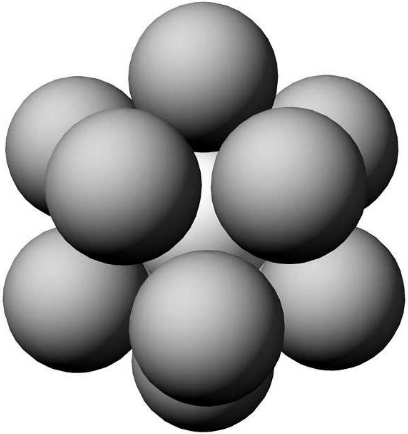
Examples of frames



(a) Non–FUNTF

(b) FUNTF

Recent applications of FUNTFs



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The geometry of finite tight frames

- We saw the vertices of platonic solids are FUNTFs.
- However, points that constitute FUNTFs do not have to be equidistributed, e.g., ONBs and Grassmanian frames.
- FUNTFs can be characterized as minimizers of a frame potential function (with Fickus) analogous to Coulomb's Law.
- Frame potential energy optimization has basic applications dealing with classification problems for hyperspectral and multi-spectral (biomedical) image data.

Frame force and potential energy

$$F : S^{d-1} \times S^{d-1} \setminus D \longrightarrow \mathbb{R}^d$$

$$P : S^{d-1} \times S^{d-1} \setminus D \longrightarrow \mathbb{R},$$

where $P(a, b) = p(\|a - b\|), \qquad p'(x) = -xf(x)$

- Coulomb force

$$CF(a, b) = (a - b)/\|a - b\|^3, \quad f(x) = 1/x^3$$

- Frame force

$$FF(a, b) = \langle a, b\rangle(a - b), \quad f(x) = 1 - x^2/2$$

- Total potential energy for the frame force

$$TFP(\{x_n\}) = \sum_{m=1}^{N}\sum_{n=1}^{N} |\langle x_m, x_n\rangle|^2$$

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Characterization of FUNTFs

Theorem

Let $N \leq d$. The minimum value of *TFP*, for the frame force and N variables, is N; and the *minimizers* are precisely the orthonormal sets of N elements for $\mathbb{R}^d$.

Let $N \geq d$. The minimum value of *TFP*, for the frame force and N variables, is N^2/d; and the *minimizers* are precisely the FUNTFs of N elements for $\mathbb{R}^d$.

Problem

Find FUNTFs analytically, effectively, computationally.

Balayage, Fourier frames, and sampling theory

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Fourier frames, goal, and a litany of names

Definition

E $= \{x_n\} \subseteq \mathbb{R}^d, \Lambda \subseteq \widehat{\mathbb{R}}^d$. E is a *Fourier frame* for $L^2(\Lambda)$ if

$\exists A, B > 0, \forall F \in L^2(\Lambda),$

$$A \, ||F||^2_{L^2(\Lambda)} \leq \sum_n | < F(\gamma), e^{-2\pi i x_n \cdot \gamma} > |^2 \leq B \, ||F||^2_{L^2(\Lambda)}.$$

- *Goal* Formulate a general theory of Fourier frames and non-uniform sampling formulas parametrized by the space $M(\mathbb{R}^d)$ of bounded Radon measures.
- *Motivation* Beurling theory (1959-1960).
- *Names* Riemann-Weber, Dini, G.D. Birkhoff, Paley-Wiener, Levinson, Duffin-Schaeffer, Beurling-Malliavin, Beurling, H.J. Landau, Jaffard, Seip, Ortega-Certà–Seip.

Balayage

- Let $M(G)$ be the algebra of bounded Radon measures on the LCAG G.
- Balayage in potential theory was introduced by Christoffel (early 1870s) and Poincaré (1890).

Definition

(Beurling) Balayage is possible for $(E, \Lambda) \subseteq G \times \widehat{G}$, a LCAG pair, if

$$\forall \mu \in M(G), \exists \nu \in M(E) \text{ such that } \hat{\mu} = \hat{\nu} \text{ on } \Lambda.$$

We write balayage (E, Λ).

- The set, Λ, of group characters is the analogue of the original role of Λ in balayage as a collection of potential theoretic kernels.
- Kahane formulated balayage for the harmonic analysis of restriction algebras.

Spectral synthesis

Definition

(Wiener, Beurling) Closed $\Lambda \subseteq \widehat{G}$ is a set of *spectral synthesis* (S-set) if
$\forall \mu \in M(G), \forall f \in C_b(G)$,
supp$(\widehat{f}) \subseteq \Lambda$ and $\hat{\mu} = 0$ on $\Lambda \Longrightarrow \int_G f \, d\mu = 0$.

($\forall T \in A'(\widehat{G}), \forall \phi \in A(\widehat{G})$, supp$(T) \subseteq \Lambda$ and $\phi = 0$ on $\Lambda \Rightarrow T(\phi) = 0$.)

- Ideal structure of $L^1(G)$ - the Nullstellensatz of harmonic analysis
- $T \in D'(\widehat{\mathbb{R}}^d), \phi \in C_c^\infty(\widehat{\mathbb{R}}^d)$, and $\phi = 0$ on supp$(T) \Rightarrow T(\phi) = 0$, with same result for $M(\widehat{\mathbb{R}}^d)$ and $C_0(\widehat{\mathbb{R}}^d)$.
- $S^2 \subseteq \widehat{\mathbb{R}}^3$ is not an S-set (L. Schwartz), and every non-discrete $\widehat{G}$ has non-S-sets (Malliavin).
- Polyhedra are S-sets. The $\frac{1}{3}$-Cantor set is an S-set with non-S-subsets.

Strict multiplicity

Definition

$\Gamma \subseteq \widehat{G}$ is a set of *strict multiplicity* if

$$\exists\ \mu \in M(\Gamma)\backslash\{0\} \text{ such that } \check{\mu} \text{ vanishes at infinity in } G.$$

- Riemann and sets of uniqueness in the wide sense.
- Menchov (1916): $\exists$ closed $\Gamma \subseteq \widehat{\mathbb{R}/\mathbb{Z}}$ and $\mu \in M(\Gamma)\backslash\{0\}$,
$$|\Gamma| = 0 \text{ and } \check{\mu}(n) = O((\log|n|)^{-1/2}), |n| \to \infty.$$
- 20th century history to study rate of decrease: Bary (1927), Littlewood (1936), Salem (1942, 1950), Ivašev-Mucatov (1957), Beurling.

Assumption

$\forall\ \gamma \in \Lambda$ and $\forall\ N(\gamma)$, compact neighborhood, $\Lambda \cap N(\gamma)$ is a set of strict multiplicity.

A theorem of Beurling

Definition

E $= \{x_n\} \subseteq \mathbb{R}^d$ is *separated* if

$$\exists\, r > 0,\ \forall m, n,\ m \neq n \Rightarrow ||x_m - x_n|| \geq r.$$

Theorem

Let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be a compact S-set, symmetric about $0 \in \widehat{\mathbb{R}}^d$, and let E $\subseteq \mathbb{R}^d$ be separated. If balayage (E,Λ), then

$$\textit{E is a Fourier frame for } L^2(\Lambda).$$

- Equivalent formulation in terms of

$$PW_\Lambda = \{f \in L^2(\mathbb{R}^d) : \mathrm{supp}(\hat{f}) \subseteq \Lambda\}.$$

- $\forall F \in L^2(\Lambda), \qquad F = \sum_{x \in \mathsf{E}} < F, S^{-1}(e_x) >_\Lambda e_x$ in $L^2(\Lambda)$.
- For $\mathbb{R}^d$ and other generality beyond Beurling's theorem in $\mathbb{R}$, the result above was formulated by Hui-Chuan Wu and JB (1998), see Landau (1967).

Lower frame bounds

- Let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be a compact S-set, and assume balayage (E, Λ) where E $= \{x_n\}$ is separated.

1. $\forall F \in L^2(\Lambda),\ \Lambda$ convex,

$$\sqrt{A}\ \frac{\int_\Lambda |F(\gamma)+F(2\gamma)+F(3\gamma)|^2\ d\gamma}{(\int_\Lambda |F(\gamma)|^2\ d\gamma)^{1/2}}$$

$$\leq (\sum |\check{F}(x_n)|^2)^{1/2} + \tfrac{1}{2}(\sum |\check{F}(\tfrac{1}{2}x_n)|^2)^{1/2} + \tfrac{1}{3}(\sum |\check{F}(\tfrac{1}{3}x_n)|^2)^{1/2}.$$

2. Given positive $G \in L^2(\Lambda)$. Then $\forall F \in L^2(\Lambda)$,

$$\sqrt{A}\ \frac{\int_\Lambda |F(\gamma)|^2 G(\gamma)\ d\gamma}{(\int_\Lambda |F(\gamma)|^2\ d\gamma)^{\frac{1}{2}}} \leq (\sum |(FG)\check{}(x_n)|^2)^{1/2}.$$

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Semi-discrete Gabor frames

Let $G \in L^2(\widehat{\mathbb{R}}^d)$ satisfy $||G||_{L^2(\widehat{\mathbb{R}}^d)} = 1$; let $\Lambda \subset \widehat{\mathbb{R}}^d$ be an S-set, symmetric about 0; and let $E \subset \mathbb{R}^d$ be separated. Define

(STFT) $\quad \forall F \in L^2(\Lambda), \quad V_G F(x,\gamma) = \int_\Lambda F(\lambda) G(\lambda - \gamma) e^{2\pi i x \cdot \lambda} \, d\lambda.$

Theorem

If balayage (E,Λ), then

$$\exists A, B > 0, \quad \forall F \in L^2(\Lambda),$$

$$A \, ||F||^2_{L^2(\Lambda)} \le \int_{\widehat{\mathbb{R}}^d} \sum_{x \in E} |V_G F(x,\gamma)|^2 \, d\gamma \le B \, ||F||^2_{L^2(\Lambda)}.$$

Remark There are basic problems to be resolved and there have been fundamental recent advances.

Examples of balayage

1. Let E $\subseteq \mathbb{R}^d$ be separated. Define

$$r = r(E) = \sup_{x \in \mathbb{R}^d} \text{dist}(x, E).$$

If $r\rho < \frac{1}{4}$, then balayage (E, $\bar{B}(0,\rho)$). $\frac{1}{4}$ is the best possible.

2. If balayage (E, Λ) and $\Lambda_0 \subseteq \Lambda$, then balayage (E, Λ_0).
3. Let E $= \{x_n\}$ be a Fourier frame for PW$_\Lambda$. Then for all $\Lambda_0 \subseteq \Lambda$ with dist$(\Lambda_0, \Lambda^c\ ) > 0$, we have balayage (E, Λ_0).
4. In $\mathbb{R}^1$, for a separated set E, Beurling lower density $> \rho$ is necessary and sufficient for balayage (E, $[\frac{-\rho}{2}, \frac{\rho}{2}]$).

Remark In $\mathbb{R}^1$, if E is uniformly dense in the sense of Duffin-Schaeffer, then $D^-(E), D^+(E)$, and $D_u(E)$ coincide.
So Beurling's result $\Rightarrow$ Duffin-Schaeffer's result on Fourier frames.

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Sampling formulas (1)

- Let $\Lambda \in \widehat{\mathbb{R}}^d$ be a compact S-set, and assume balayage (E, Λ), E $= \{x_n\} \subseteq \mathbb{R}^d$ separated.
- *Theorem* $\exists \epsilon > 0$, balayage (E, Λ_ϵ).
- *Theorem* $\forall x \in \mathbb{R}^d,\ \exists\ \{b_n(x)\} \in l^1(\mathbb{Z})$,

$$\sup_{x\in\mathbb{R}^d} \sum_n |b_n(x)| \leq K(E, \Lambda_\epsilon)$$

and $$e^{-2\pi i x\cdot\gamma} = \sum_n b_n(x) e^{-2\pi i x_n\cdot\gamma}$$ uniformly on Λ_ϵ.

- Let h be entire on $\mathbb{R}^d$ with $e^{-\Omega(|x|)}$ decay,

$$h(0) = 1 \text{ and } \mathrm{supp}(\hat{h}) \subseteq \bar{B}(0, \epsilon).$$

Theorem

$\forall f \in C_b(\mathbb{R})$, *supp*$(\hat{f}) \subseteq \Lambda$,

$$\forall y \in \mathbb{R}^d, \quad f(y) = \sum f(x_n) b_n(y) h(x_n - y)$$

- Weighted sampling function $b_n(y)h(x_n - y)$ independent of $f \in C_b(\mathbb{R}^d)$, supp$(\hat{f}) \subseteq \Lambda$.

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Sampling formulas (2)

- The Nyquist condition, $2T\Omega \leq 1$, for sampling period T and bandwidth $[-\Omega, \Omega]$, gives way to balayage (E,Λ), where Λ is the bandwidth and the sampling set E is related to Λ by balayage (E,Λ).
- Let $s \in C_b(\mathbb{R}^d)$, supp$(\hat{s}) \subseteq \Lambda$, a compact S-set - *sampling function* s.
- Let $A = \{a(n)\} \subseteq \mathbb{R}^d, n \in \mathbb{Z}$ and distinct points $a(n)$. Define

$$V_A = \{f \in C_b(\mathbb{R}^d) : \forall x \in \mathbb{R}^d, f(x) = \sum_n c_n(f)s(x-a(n)), \sum_n |c_n(f)| < \infty\}.$$

- Assume balayage (E,Λ), E $= \{x_n\} \subseteq \mathbb{R}^d$ separated.
- Define

$$V_E = \{f \in C_b(\mathbb{R}^d) : \forall x \in \mathbb{R}^d, f(x) = \sum_n c_n(f)s(x-x_n), \sum_n |c_n(f)| < \infty\}.$$

Theorem

$V = \cup_A V_A \subseteq V_E \subseteq C_b(\mathbb{R}^d)$. *Thus,*

$$\forall f \in V, \quad f(x) = \textstyle\sum_n c_n(f)s(x-x_n), \text{ uniformly on } \mathbb{R}^d.$$

- Signal decomposition in terms of (E, Λ)-balayage, defined by measures whose absolutely convergent non-harmonic Fourier series are generalized characters parameterized by Λ.
- Sampling multipliers and lower frame bound inequalities
- Pseudo-differential operator sampling formulas
- Bilinear frame operators and classical extensions of the Calderon formula in harmonic analysis

That's all folks!

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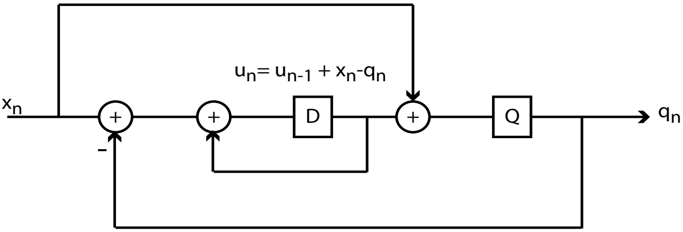
Quantization Methods

SIGMA-DELTA QUANTIZATION

Given u_0 and $\{x_n\}_{n=1}$

$u_n = u_{n-1} + x_n - q_n$
$q_n = Q(u_{n-1} + x_n)$



First Order ΣΔ

A quantization problem

Qualitative Problem Obtain *digital* representations for class X, suitable for storage, transmission, recovery.
Quantitative Problem Find dictionary $\{e_n\} \subseteq X$:

1. Sampling [continuous range $\mathbb{K}$ is not digital]

$$\forall x \in X, \quad x = \sum x_n e_n, \quad x_n \in \mathbb{K}.$$

2. Quantization. Construct finite alphabet $\mathcal{A}$ and

$$Q: \; X \to \{\sum q_n e_n : \; q_n \in \mathcal{A} \subseteq \mathbb{K}\}$$

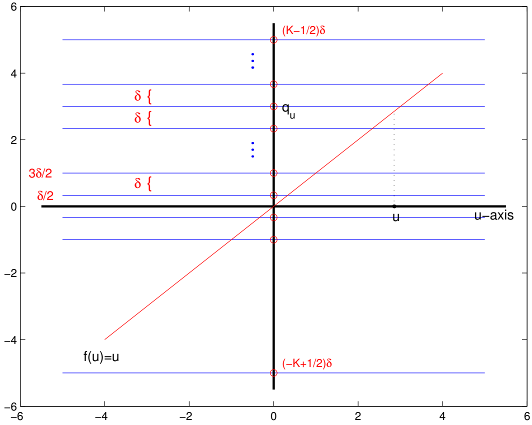
such that $|x_n - q_n|$ and/or $\|x - Qx\|$ small.

Methods

Fine quantization, e.g., PCM. Take $q_n \in \mathcal{A}$ close to given x_n. Reasonable in 16-bit (65,536 levels)digital audio.
Coarse quantization, e.g., $\Sigma\Delta$. Use fewer bits to exploit redundancy.
SRQP

Quantization

$$\mathcal{A}_K^\delta = \{(-K+1/2)\delta, (-K+3/2)\delta, \ldots, (-1/2)\delta, (1/2)\delta, \ldots, (K-1/2)\delta\}$$



$$Q(u) = \arg\min\{|u-q| : q \in \mathcal{A}_K^\delta\} = q_u$$

PCM

Replace $x_n \leftrightarrow q_n = \arg\{\min |x_n - q| : q \in \mathcal{A}_K^\delta\}$. Then

$$(PCM) \qquad \tilde{x} = \frac{d}{N} \sum_{n=1}^{N} q_n e_n$$

satisfies

$$\|x - \tilde{x}\| \leq \frac{d}{N} \| \sum_{n=1}^{N} (x_n - q_n) e_n \| \leq \frac{d}{N} \frac{\delta}{2} \sum_{n=1}^{N} \|e_n\| = \frac{d}{2} \delta.$$

Not good!

Bennett's white noise assumption

Assume that $(\eta_n) = (x_n - q_n)$ is a sequence of independent, identically distributed random variables with mean 0 and variance $\frac{\delta^2}{12}$. Then the mean square error (MSE) satisfies

$$\text{MSE} = E\|x - \tilde{x}\|^2 \leq \frac{d}{12A} \delta^2 = \frac{(d\delta)^2}{12N}$$

$\mathcal{A}_1^2 = \{-1, 1\}$ and E_7

Let $x = (\frac{1}{3}, \frac{1}{2})$, $E_7 = \{(\cos(\frac{2n\pi}{7}), \sin(\frac{2n\pi}{7}))\}_{n=1}^{7}$. Consider quantizers with $\mathcal{A} = \{-1, 1\}$.

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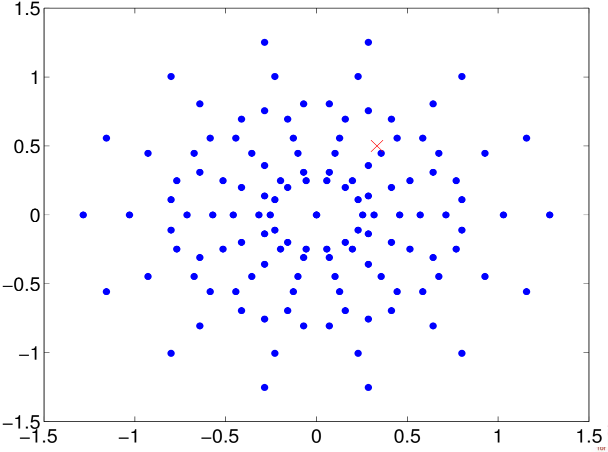

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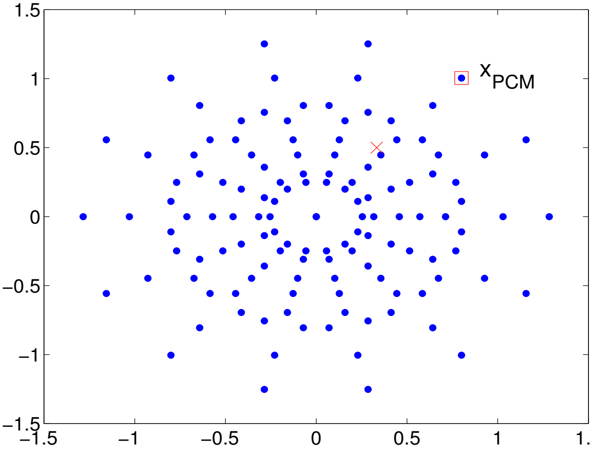

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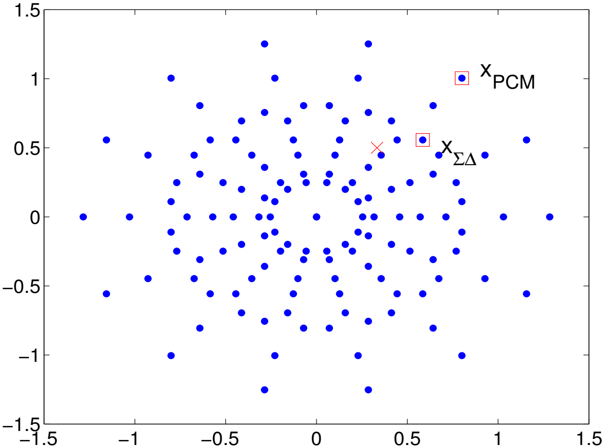

Sigma-Delta quantization – background

- History from 1950s.
- Treatises of Candy, Temes (1992) and Norsworthy, Schreier, Temes (1997).
- PCM for finite frames and $\Sigma\Delta$ for PW_Ω:
  Bölcskei, Daubechies, DeVore, Goyal, Güntürk, Kovačevič, Thao, Vetterli.
- Combination of $\Sigma\Delta$ and finite frames:
  Powell, Yılmaz, and B.
- Subsequent work based on this $\Sigma\Delta$ finite frame theory:
  Bodman and Paulsen; Boufounos and Oppenheim; Jimenez and Yang Wang; Lammers, Powell, and Yılmaz.
- Genuinely apply it.

PCM and first order Sigma-Delta

Let $x \in \mathbb{C}^d$, $\{e_n\}_{n=1}^N$ be a frame for $\mathbb{C}^d$.

- **PCM:** $\forall n = 1, \ldots, N, \quad q_n = Q_\delta(\langle x, e_n \rangle)$,
- **First Order Sigma-Delta:** Let p be a permutation of $\{1, \ldots, N\}$. First Order Sigma-Delta quantization generates quantized sequence $\{q_n\}_{n=1}^N$ by the iteration

$$\begin{aligned} q_n &= Q_\delta(u_{n-1} + \langle x, e_{p(n)} \rangle), \\ u_n &= u_{n-1} + \langle x, e_{p(n)} \rangle - q_n, \end{aligned}$$

for $n = 1, \ldots, N$, with an initial condition u_0.

In either case, the quantized estimate is

$$\tilde{x} = \frac{d}{N} \sum_{n=1}^{N} q_n e_n = \frac{d}{N} L^* q$$

$\Sigma\Delta$ quantizers for finite frames

Let $F = \{e_n\}_{n=1}^N$ be a frame for $\mathbb{R}^d$, $x \in \mathbb{R}^d$.

Define $x_n = \langle x, e_n \rangle$.

Fix the ordering p, a permutation of $\{1, 2, \ldots, N\}$.

Quantizer alphabet $\mathcal{A}_K^\delta$

Quantizer function $Q(u) = \arg\{\min |u - q| : q \in \mathcal{A}_K^\delta\}$

Define the *first-order* $\Sigma\Delta$ *quantizer* with ordering p and with the quantizer alphabet $\mathcal{A}_K^\delta$ by means of the following recursion.

$$\begin{aligned} u_n - u_{n-1} &= x_{p(n)} - q_n \\ q_n &= Q(u_{n-1} + x_{p(n)}) \end{aligned}$$

where $u_0 = 0$ and $n = 1, 2, \ldots, N$.

Stability

The following stability result is used to prove error estimates.

Proposition

If the frame coefficients $\{x_n\}_{n=1}^N$ satisfy

$$|x_n| \le (K - 1/2)\delta, \quad n = 1, \cdots, N,$$

then the state sequence $\{u_n\}_{n=0}^N$ generated by the first-order $\Sigma\Delta$ quantizer with alphabet $\mathcal{A}_K^\delta$ satisfies $|u_n| \le \delta/2, n = 1, \cdots, N.$

- The first-order $\Sigma\Delta$ scheme is equivalent to

$$u_n = \sum_{j=1}^{n} x_{p(j)} - \sum_{j=1}^{n} q_j, \quad n = 1, \cdots, N.$$

- Stability results lead to tiling problems for higher order schemes.

Error estimate

Definition

Let $F = \{e_n\}_{n=1}^N$ be a frame for $\mathbb{R}^d$, and let p be a permutation of $\{1, 2, \ldots, N\}$. The *variation* $\sigma(F, p)$ is

$$\sigma(F, p) = \sum_{n=1}^{N-1} \|e_{p(n)} - e_{p(n+1)}\|.$$

Error estimate

Theorem

Let $F = \{e_n\}_{n=1}^N$ be an A-FUNTF for $\mathbb{R}^d$. The approximation

$$\tilde{x} = \frac{d}{N}\sum_{n=1}^{N} q_n e_{p(n)}$$

generated by the first-order $\Sigma\Delta$ quantizer with ordering p and with the quantizer alphabet $\mathcal{A}_K^\delta$ satisfies

$$\|x - \tilde{x}\| \leq \frac{(\sigma(F,p)+1)d}{N}\,\frac{\delta}{2}.$$

Harmonic frames

Zimmermann and Goyal, Kelner, Kovačević, Thao, Vetterli.

Definition

$H = \mathbb{C}^d$. An *harmonic frame* $\{e_n\}_{n=1}^N$ for H is defined by the rows of the Bessel map L which is the complex N-DFT $N \times d$ matrix with $N - d$ columns removed.

$H = \mathbb{R}^d$, d even. The harmonic frame $\{e_n\}_{n=1}^N$ is defined by the Bessel map L which is the $N \times d$ matrix whose nth row is

$$e_n^N = \sqrt{\frac{2}{d}}\left(\cos(\frac{2\pi n}{N}), \sin(\frac{2\pi n}{N}), \dots, \cos(\frac{2\pi(d/2)n}{N}), \sin(\frac{2\pi(d/2)n}{N})\right).$$

- Harmonic frames are FUNTFs.
- Let E_N be the harmonic frame for $\mathbb{R}^d$ and let p_N be the identity permutation. Then

$$\forall N,\ \sigma(E_N, p_N) \le \pi d(d+1).$$

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Error estimate for harmonic frames

Theorem

Let E_N be the harmonic frame for $\mathbb{R}^d$ with frame bound N/d. Consider $x \in \mathbb{R}^d$, $\|x\| \leq 1$, and suppose the approximation $\tilde{x}$ of x is generated by a first-order $\Sigma\Delta$ quantizer as before. Then

$$\|x - \tilde{x}\| \leq \frac{d^2(d+1)+d}{N}\,\frac{\delta}{2}.$$

- Hence, for harmonic frames (and all those with bounded variation),

$$\mathrm{MSE}_{\Sigma\Delta} \leq \frac{C_d}{N^2}\,\delta^2.$$

- This bound is clearly superior asymptotically to

$$\mathrm{MSE}_{\mathrm{PCM}} = \frac{(d\delta)^2}{12N}.$$

$\Sigma\Delta$ and "optimal" PCM

Theorem

The first order $\Sigma\Delta$ scheme achieves the asymptotically optimal $\mathrm{MSE}_{\mathrm{PCM}}$ for harmonic frames.

The digital encoding

$$\mathrm{MSE}_{\mathrm{PCM}} = \frac{(d\delta)^2}{12N}$$

in PCM format leaves open the possibility that decoding (consistent nonlinear reconstruction, with additional numerical complexity this entails) could lead to

$$\text{“}\mathrm{MSE}^{\mathrm{opt}}_{\mathrm{PCM}}\text{”} \ll O(\frac{1}{N}).$$

Goyal, Vetterli, Thao (1998) proved

$$\text{“}\mathrm{MSE}^{\mathrm{opt}}_{\mathrm{PCM}}\text{”} \sim \frac{\tilde{C}_d}{N^2}\delta^2.$$

A comparison of Σ-Δ and PCM

Comparison of 1-bit PCM and 1-bit $\Sigma\Delta$

Let $x \in \mathbb{C}^d$, $||x|| \leq 1$.

Definition

- $q_{PCM}(x)$ is the sequence to which x is mapped by PCM.
- $q_{\Sigma\Delta}(x)$ is the sequence to which x is mapped by $\Sigma\Delta$.
- $$\mathrm{err}_{PCM}(x) = ||x - \frac{d}{N} L^* q_{PCM}(x)||$$
  $$\mathrm{err}_{\Sigma\Delta}(x) = ||x - \frac{d}{N} L^* q_{\Sigma\Delta}(x)||$$

Fickus question: We shall analyze to what extent $\mathrm{err}_{\Sigma\Delta}(x) < \mathrm{err}_{PCM}(x)$ beyond our results with Powell and Yilmaz.

PCM and first order Sigma-Delta

Let $x \in \mathbb{C}^d$,
Let $F = \{e_n\}_{n=1}^N$ be a FUNTF for $\mathbb{C}^d$ with the analysis matrix L.

Definition

- $q_{PCM}(x, F, b)$ is the quantized sequence given by b-bit PCM,
- $q_{\Sigma\Delta}(x, F, b)$ is the quantized sequence given by b-bit Sigma-Delta.
- $$\mathtt{err}_{PCM}(x, F, b) = \|x - \frac{d}{N} L^* q_{PCM}(x)\|,$$ $$\mathtt{err}_{\Sigma\Delta}(x, F, b) = \|x - \frac{d}{N} L^* q_{\Sigma\Delta}(x)\|.$$

Comparison of 1-bit PCM and 1-bit $\Sigma\Delta$

Definition

A function $e : [a, b] \to \mathbb{C}^d$ is of *bounded variation (BV)* if there is a $K > 0$ such that for every $a \le t_1 < t_2 < \cdots < t_N \le b$,

$$\sum_{n=1}^{N-1} \|e(t_n) - e(t_{n+1})\| \le K.$$

The smallest such K is denoted by $|e|_{BV}$, and defines a seminorm for the space of BV functions.

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Comparison of 1-bit PCM and 1-bit Sigma-Delta

Theorem 1

Let $x \in \mathbb{C}^d$ satisfy $0 < \|x\| \leq 1$, and let $F = \{e_n\}_{n=1}^N$ be a FUNTF for $\mathbb{C}^d$. Then, the 1-bit PCM error satisfies

$$\mathtt{err}_{PCM}(x, F, 1) \geq \alpha_F + 1 - \|x\|$$

where

$$\alpha_F := \inf_{\|x\|=1} \frac{d}{N} \sum_{n=1}^{N} \left(|Re(\langle x, e_n\rangle)| + |Im(\langle x, e_n\rangle)|\right) - 1 \geq 0.$$

Comparison of 1-bit PCM and 1-bit Sigma-Delta

Theorem 2

Let $\{F_N = \{e_n^N\}_{n=1}^N\}$ be a family of FUNTFs for $\mathbb{C}^d$. Then,

$$\forall \varepsilon > 0,\ \exists N_0 > 0,\ \textit{ such that }\ \forall N \geq N_0\ \textit{ and }\ \forall 0 < \|x\| \leq 1 - \varepsilon$$

$$\texttt{err}_{\Sigma\Delta}(x, F_N, 1) \leq \texttt{err}_{PCM}(x, F_N, 1).$$

Numerical experiments suggest that, we can choose N significantly smaller than $(M/\varepsilon)^{2d}$.

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Comparison of 1-bit PCM and 1-bit Sigma-Delta

If $\{\alpha_{F_N}\}$ is bounded below by a positive number, then we can improve Theorem 2:

Theorem 3

Let $\{F_N = \{e_n^N\}_{n=1}^N\}$ be a family of FUNTFs for $\mathbb{C}^d$ such that

$$\exists a > 0,\ \forall N,\ \alpha_{F_N} \geq a.$$

Then,

$$\exists N_0 > 0 \ \textit{ such that } \ \forall N \geq N_0 \ \textit{ and } \ \forall 0 < \|x\| \leq 1$$

$$\mathtt{err}_{\Sigma\Delta}(x, F_N, 1) \leq \mathtt{err}_{PCM}(x, F_N, 1).$$

Comparison of 1-bit PCM and 1-bit Sigma-Delta

Below is a family $\{F_N\}$ of FUNTFs where $\{\alpha_{F_N}\}$ is bounded below by a positive constant. Harmonic frames are examples to such families.

Theorem 4

Let $e : [0,1] \rightarrow \{x \in \mathbb{C}^d : \|x\| = 1\}$ be continuous function of bounded variation such that $F_N = \{e(n/N)\}_{n=1}^N$ is a FUNTF for $\mathbb{C}^d$ for every N. Then,

$$\exists N_0 > 0 \textit{ such that } \forall N \geq N_0 \textit{ and } \forall 0 < \|x\| \leq 1$$

$$\mathtt{err}_{\Sigma\Delta}(x, F_N, 1) \leq \mathtt{err}_{PCM}(x, F_N, 1).$$

One can show that $\alpha := \lim_{N\to\infty} \alpha_{F_N}$ is positive, and that

$$\alpha + 1 = d \inf_{\|x\|=1} \int_0^1 \left(|Re(\langle x, e(t)\rangle)| + |Im(\langle x, e(t)\rangle)|\right) dt.$$

Comparison of 1-bit PCM and 1-bit $\Sigma\Delta$

Theorem

Let $e : [0,1] \to \{x \in \mathbb{C}^d : \|x\| = 1\}$ be continuous function of bounded variation such that $F_N = (e(n/N))_{n=1}^N$ is a FUNTF for $\mathbb{C}^d$ for every N. Then,

$$\exists N_0 > 0 \textit{ such that } \forall N \geq N_0 \textit{ and } \forall 0 < \|x\| \leq 1$$

$$\mathtt{err}_{\Sigma\Delta}(x) \leq \mathtt{err}_{PCM}(x).$$

Moreover, a lower bound for N_0 is $d(1 + |e|_{BV})/(\sqrt{d} - 1)$.

Comparison of 1-bit PCM and 1-bit $\Sigma\Delta$

Example (Roots of unity frames for $\mathbb{R}^2$)

$e_n^N = (\cos(2\pi n/N), \sin(2\pi n/N))$.

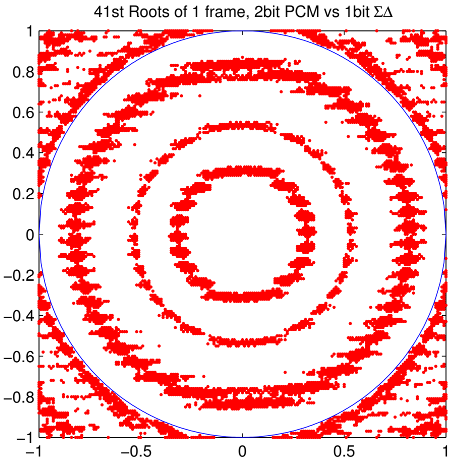
Here, $e(t) = (\cos(2\pi t), \sin(2\pi t))$,
$M = |e|_{BV} = 2\pi$, $lim \alpha_{F_N} = 2/\pi$.

Example (Real Harmonic Frames for $\mathbb{R}^{2k}$)

$e_n^N = \frac{1}{\sqrt{k}}(\cos(2\pi n/N), \sin(2\pi n/N), \ldots, \cos(2\pi kn/N), \sin(2\pi kn/N))$.

In this case, $e(t) = \frac{1}{\sqrt{k}}(\cos(2\pi t), \sin(2\pi t), \ldots, \cos(2\pi kt), \sin(2\pi kt))$,
$M = |e|_{BV} = 2\pi\sqrt{\frac{1}{d}\sum_{k=1}^{d} k^2}$.

Comparison of 2-bit PCM and 1-bit ΣΔ



Red: $\mathtt{err}_{PCM}(x) < \mathtt{err}_{\Sigma\Delta}(x)$, Green: $\mathtt{err}_{PCM}(x) = \mathtt{err}_{\Sigma\Delta}(x)$.

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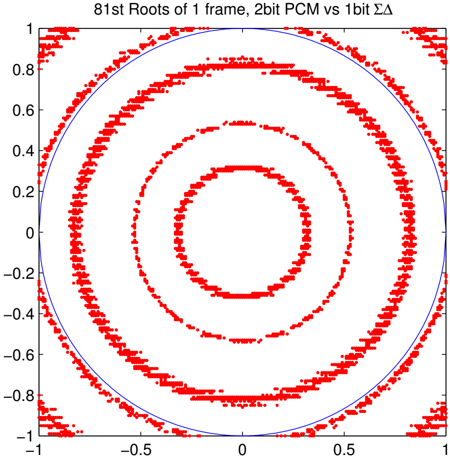
Comparison of 2-bit PCM and 1-bit ΣΔ



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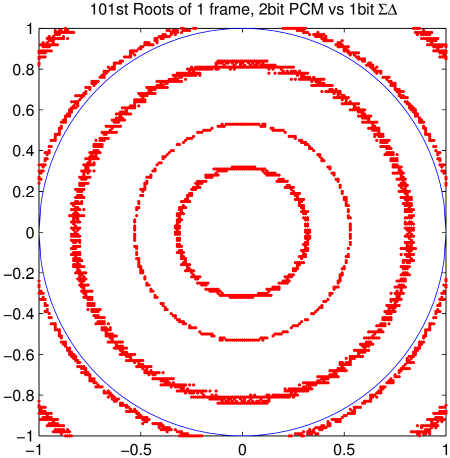
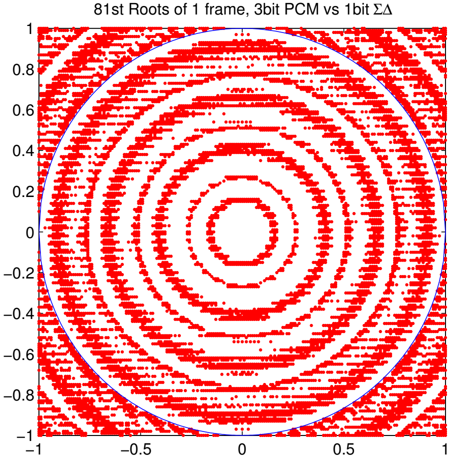
Comparison of 2-bit PCM and 1-bit ΣΔ



Red: $\mathtt{err}_{PCM}(x) < \mathtt{err}_{\Sigma\Delta}(x)$, Green: $\mathtt{err}_{PCM}(x) = \mathtt{err}_{\Sigma\Delta}(x)$.

Comparison of 3-bit PCM and 1-bit $\Sigma\Delta$



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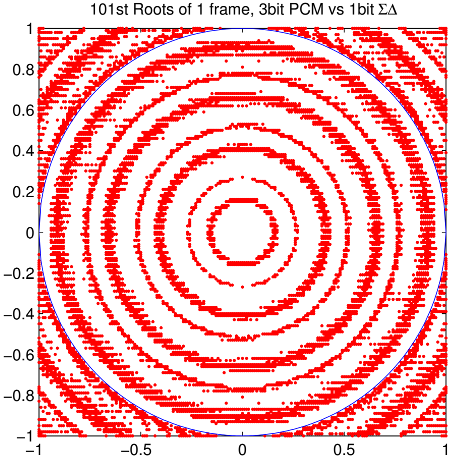
Comparison of 3-bit PCM and 1-bit $\Sigma\Delta$



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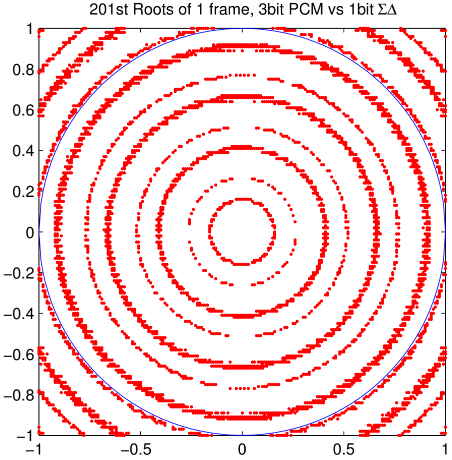
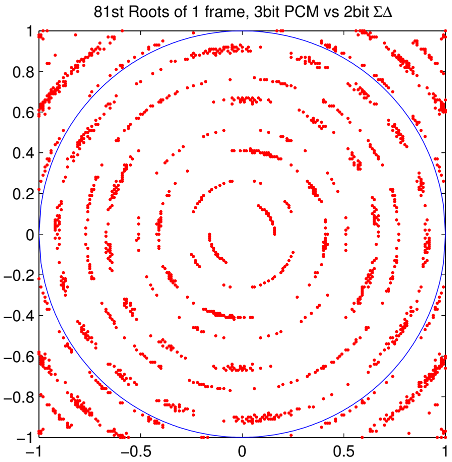
Comparison of 3-bit PCM and 1-bit ΣΔ



Red: $\mathtt{err}_{PCM}(x) < \mathtt{err}_{\Sigma\Delta}(x)$, Green: $\mathtt{err}_{PCM}(x) = \mathtt{err}_{\Sigma\Delta}(x)$.

Comparison of 3-bit PCM and 2-bit $\Sigma\Delta$



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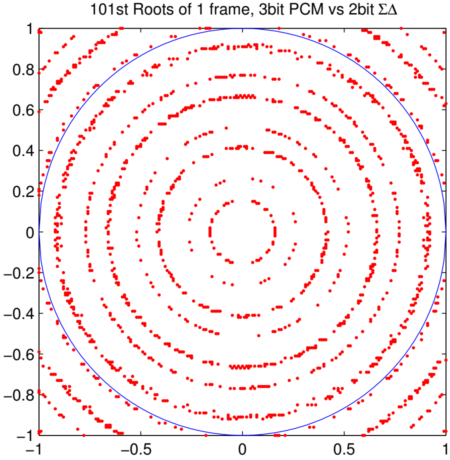
Comparison of 3-bit PCM and 2-bit $\Sigma\Delta$



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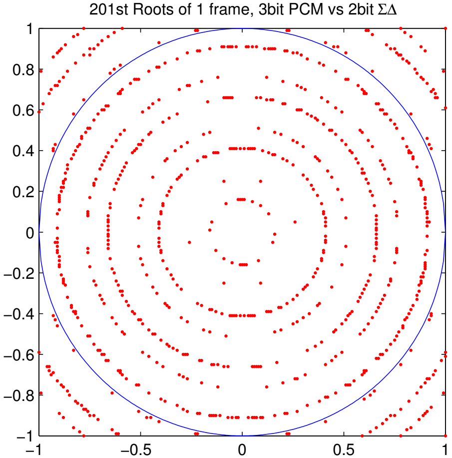
Comparison of 3-bit PCM and 2-bit $\Sigma\Delta$



Red: $\mathtt{err}_{PCM}(x) < \mathtt{err}_{\Sigma\Delta}(x)$, Green: $\mathtt{err}_{PCM}(x) = \mathtt{err}_{\Sigma\Delta}(x)$.

Complex Σ-Δ and Yang Wang's idea and algorithm

Complex ΣΔ - Alphabet

Let $K \in \mathbb{N}$ and $\delta > 0$. The *midrise* quantization alphabet is

$$\mathcal{A}_K^\delta = \left\{ \left(m + \frac{1}{2} \right) \delta + in\delta : m = -K, \ldots, K-1, \ n = -K, \ldots, K \right\}$$

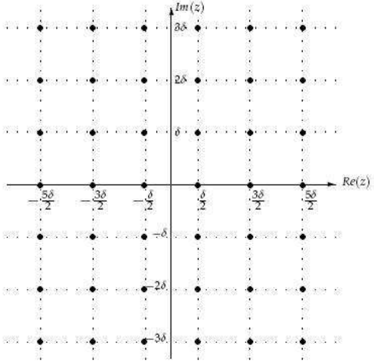


Figure: $\mathcal{A}_K^\delta$ for $K = 3\delta$.

Alphabet

For $K > 0$ (we consider only $K = 1$)
and $b \geq 1$, an integer representing the number of bits,
let $\delta = 2K/(2^b - 1)$.

$$\mathcal{A}_\delta^K = \{(-K + m\delta) + i(-K + n\delta) : m, n = 0, \ldots, 2^b - 1\}.$$

The associated *scalar uniform quantizer* is

$$Q_\delta(u + iv) = \delta\left(\frac{1}{2} + \left\lfloor\frac{u}{\delta}\right\rfloor + i(\frac{1}{2} + \left\lfloor\frac{v}{\delta}\right\rfloor)\right).$$

In particular, for 1-bit case, $Q(u + iv) = sign(u) + isign(v)$

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Complex $\Sigma\Delta$

The *scalar uniform quantizer* associated to $\mathcal{A}_K^\delta$ is

$$Q_\delta(a+ib) = \delta\left(\frac{1}{2} + \left\lfloor\frac{a}{\delta}\right\rfloor + i\left\lfloor\frac{b}{\delta}\right\rfloor\right),$$

where $\lfloor x \rfloor$ is the largest integer smaller than x.
For any $z = a + ib$ with $|a| \leq K$ and $|b| \leq K$, Q satisfies

$$|z - Q_\delta(z)| \leq \min_{\zeta \in \mathcal{A}_K^\delta} |z - \zeta|.$$

Let $\{x_n\}_{n=1}^N \subseteq \mathbb{C}$ and let p be a permutation of $\{1, \ldots, N\}$. Analogous to the real case, the first order $\Sigma\Delta$ quantization is defined by the iteration

$$\begin{aligned} u_n &= u_{n-1} + x_{p(n)} - q_n, \\ q_n &= Q_\delta(u_{n-1} + x_{p(n)}). \end{aligned}$$

Complex $\Sigma\Delta$

The following theorem is analogous to BPY

Theorem

Let $F = \{e_n\}_{n=1}^N$ be a finite unit norm frame for $\mathbb{C}^d$, let p be a permutation of $\{1, \ldots, N\}$, let $|u_0| \leq \delta/2$, and let $x \in \mathbb{C}^d$ satisfy $\|x\| \leq (K - 1/2)\delta$. The $\Sigma\Delta$ approximation error $\|x - \widetilde{x}\|$ satisfies

$$\|x - \widetilde{x}\| \leq \sqrt{2}\|S^{-1}\|_{\mathrm{op}} \left(\sigma(F,p)\frac{\delta}{2} + |u_N| + |u_0|\right),$$

where S^{-1} is the inverse frame operator. In particular, if F is a FUNTF, then

$$\|x - \widetilde{x}\| \leq \sqrt{2}\frac{d}{N} \left(\sigma(F,p)\frac{\delta}{2} + |u_N| + |u_0|\right),$$

Complex $\Sigma\Delta$

Let $\{F_N\}$ be a family of FUNTFs, and p_N be a permutation of $\{1, \ldots, N\}$. Then the frame variation $\sigma(F_N, p_N)$ is a function of N. If $\sigma(F_N, p_N)$ is bounded, then

$$\|x - \widetilde{x}\| = \mathcal{O}(N^{-1}) \text{ as } N \to \infty.$$

Wang gives an upper bound for the frame variation of frames for $\mathbb{R}^d$, using the results from the Travelling Salesman Problem.

Theorem YW

Let $S = \{v_j\}_{j=1}^N \subseteq [-\frac{1}{2}, \frac{1}{2}]^d$ with $d \geq 3$. There exists a permutation p of $\{1, \ldots, N\}$ such that

$$\sum_{j=1}^{N-1} \|v_{p(j)} - v_{p(j+1)}\| \leq 2\sqrt{d+3}N^{1-\frac{1}{d}} - 2\sqrt{d+3}.$$

Complex $\Sigma\Delta$

Theorem

Let $F = \{e_n\}_{n=1}^N$ be a FUNTF for $\mathbb{R}^d$, $|u_0| \leq \delta/2$, and let $x \in \mathbb{R}^d$ satisfy $\|x\| \leq (K - 1/2)\delta$. Then, there exists a permutation p of $\{1, 2, \dots, N\}$ such that the approximation error $\|x - \widetilde{x}\|$ satisfies

$$\|x - \widetilde{x}\| \leq \sqrt{2}\delta d \left((1 - \sqrt{d+3}) N^{-1} + \sqrt{d+3} N^{-\frac{1}{d}} \right)$$

This theorem guarantees that

$$\|x - \widetilde{x}\| \leq \mathcal{O}(N^{-\frac{1}{d}}) \ as \ N \rightarrow \infty$$

for FUNTFs for $\mathbb{R}^d$.

That’s all folks!

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Preprocessing for clutter mitigation

- Massive sensor data set → dimension reduction → sparse representation
- False targets caused by clutter inhibit data triage, waste vital resources, and degrade sparse representation algorithms
- View clutter mitigation as preprocessing step for ATR/ATE
- For active sensors, choose waveform to reduce clutter effects by limiting side lobe magnitude
  - improves concise data representation
  - supports dimensionality reduction processing

Sparse coefficient sets for stable representation

- Opportunistic sensing systems can utilize large networks of diverse sensors
  - sensor quality may vary, e.g., low cost wireless sensors
  - massive amount of noisy sensor data
- Signal representations using sparse coefficient sets
  - compensate for hardware errors
  - ensure numerical stability
  - frame setting → frame dimension reduction

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Frame variation and $\Sigma\Delta$

- $F = \{e_j\}_{j=1}^N$ a FUNTF for $\mathbb{C}^d$
- $x \in \mathbb{C}^d$, p a permutation of $\{1, ..., N\}$, $x_{p(n)} = \langle x, e_{p(n)} \rangle$,

$$x = \frac{d}{N} \sum_{n=1}^{N} x_{p(n)} e_{p(n)} \text{ and } \tilde{x} \equiv \frac{d}{N} \sum_{n=1}^{N} q_n e_{p(n)}$$

- Frame variation,

$$\sigma(F, p) = \sum_{n=1}^{N-1} \left\| e_{p(n)} - e_{p(n+1)} \right\|$$

- Transport $\Sigma\Delta$ FUNTF setting to coefficient sparse representation point of view.

Summary

Given a signal x and a tolerance $r > 0$

- Define frames using Frame Potential Energy and SQP (or other optimization)
- Analyze Frame Variation in terms of our permutation algorithm
- Compute $\tilde{x}$ having separated coefficients taken from a fixed small and sparse alphabet
- Ensure that $\|x - \tilde{x}\| < r$.

Conclusion: $\tilde{x}$ is a stable sparse coefficient approximant of x

Frame potential classification algorithm

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Outline

- Retinal imaging and pre-drusen problem
- Multispectral retinal imaging
- Dimension reduction
- Frames
- Frame potential classification algorithm
- Image processing to detect pre-drusen
- Hyperspectral image processing

Dimension reduction

Kernel dimension reduction

Given data space X of N vectors in $\mathbb{R}^D$. (N is the number of pixels in the hypercube, D is the number of spectral bands.)

Two Steps:

1. Construction of an $N \times N$ symmetric, positive semi–definite kernel, K, from these N data points in $\mathbb{R}^D$.
2. Diagonalization of K, and then choosing $d \leq D$ *significant orthogonal* eigenmaps of K.

- Different classes of interest may not be orthogonal to each other; however, they may be captured by different frame elements. It is plausible that classes may correspond to elements in a frame but not elements in a basis.
- A *frame* generalizes the concept of an orthonormal basis. Frame elements are non–orthogonal.

Dimension reduction paradigm

- Given data space X of N vectors $x_m \in \mathbb{R}^D$, and let
$$K : X \times X \to \mathbb{R}$$
be a symmetric ($K(x,y) = K(y,x)$), positive semi–definite kernel.
- We map X to a low dimensional space via the following mapping:
$$X \longrightarrow K \longrightarrow \mathbb{R}^d(K), \quad d < D$$
$$x_m \mapsto y_m = (y[m,n_1], y[m,n_2], \ldots, y[m,n_d]) \in \mathbb{R}^d(K),$$
where $y[\cdot,n] \in \mathbb{R}^N$ is an eigenvector of K.

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LLE

- Consider the data points X as the nodes of a graph.
- Define a metric $\rho : X \times X \longrightarrow \mathbb{R}^+$, e.g., $\rho(x_m, x_n) = \|x_m - x_n\|$ is the Euclidean distance.
- Choose $q \in \mathbb{N}$.
- For each x_i choose the q nodes x_n closest to x_i in the metric ρ, and place an edge between x_i and each of these nodes.
- This defines $N'(x_i)$, viz.,
  $N'(x_i) = \{x \in X \ : \ \exists$ an edge between x and $x_i.\}$.
- To define the weights on the edges, we compute:

$$W = \underset{\widetilde{W}}{\operatorname{argmin}} \left| x_i - \sum_{j \in N'(x_i)} \widetilde{W}(x_i, x_j) x_j \right|^2 .$$

- Set $K = (I - W)(I - W^{\mathsf{T}})$ and diagonalize K.
- K is symmetric and positive semi–definite.

Laplacian Eigenmaps

- Consider the data points X as the nodes of a graph.
- Define a metric $\rho : X \times X \longrightarrow \mathbb{R}^+$, e.g., $\rho(x_m, x_n) = \|x_m - x_n\|$ is the Euclidean distance.
- Choose $q \in \mathbb{N}$.
- For each x_i choose the q nodes x_n closest to x_i in the metric ρ, and place an edge between x_i and each of these nodes.
- This defines $N'(x_i)$, viz.,
  $N'(x_i) = \{x \in X\ :\ \exists$ an edge between x and $x_i.\}$.
- To define the weights on the edges, we compute:

$$W_{ij} = \begin{cases} \exp(-\|x_i - x_j\|^2/\sigma) & \text{if } x_j \in N'(x_i) \text{ or } x_i \in N'(x_j) \\ 0 & \text{otherwise} \end{cases}$$

- Set $K = D - W$, where $D_{ii} = \sum_j W_{ij}$ and $D_{ij} = 0$ for $i \neq j$;
- Diagonalize K.
- K is symmetric and positive semi–definite.

Frames

Frame potential classification algorithm

Optimization problem: maximal separation

Goal: Construct a FUNTF $\{\Psi_k\}_{k=1}^s$ such that each Ψ_k is associated to only one classifiable material.

For $\{\theta_k\}_{k=1}^s \in S^{d-1} \times \cdots \times S^{d-1}$ and $n = 1, \ldots, s$, set

$$p(\theta_n) = \sum_{m=1}^{N} |\langle y_m, \theta_n \rangle|$$

and consider the maximal separation

$$\sup_{\{\theta_j\}_{j=1}^s} \min\{|p(\theta_k) - p(\theta_n)| : k \neq n\}.$$

Optimization problem: FUNTF construction

Combine maximal separation with frame potential to construct a pseudo-FUNTF $\Psi = \{\psi_k\}_{k=1}^s$ by solving the minimization problem:

$$\sup \left\{ \min\{|p(\theta_k) - p(\theta_n)| : k \neq n\} : \{\theta_j\} \in \{\arg\min_{\Phi} TFP(\Phi)\right\}, \quad (1)$$

where $\Phi = \{\phi_k\}_{k=1}^s$.

- (1) is solved using a new, fast gradient descent method for products of spheres.
- Nate Strawn created the method and developed new geometric ideas for such computation.

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Optimization problem: FUNTF construction

Combine frame potential with "ℓ^1-energy" to construct a FUNTF $\Psi = \{\psi_k\}_{k=1}^{s}$ by solving a minimization problem of the following type:

$$\min\{TFP(\Theta) + P(Y, \Theta) : \Theta \in S^{d-1} \times \cdots \times S^{d-1}\},$$

where

$$P(Y, \Theta) = \sum_{n=1}^{N} \sum_{k=1}^{s} |\langle y_n, \theta_k \rangle| = \sum_{k=1}^{s} p(\theta_k).$$

Remark. a. Minimization of P is convex optimization of ℓ^1-energy of Y for a given frame.
b. By Candes and Tao (2005), under suitable conditions, this can yield a frame Ψ with a sparse set of coefficients $\{\langle y_n, \psi_k \rangle\}$. We do not proceed this way to obtain sparsity.

Frame coefficient images

Given $\Psi = \{\Psi_n\}_{n=1}^{s}$ and $m \in \{1, \ldots, N\}$. Consider the set of frame decompositions

$$y_m = \sum_{n=1}^{s} c_{m,n}^{\alpha} \Psi_n, \quad \text{indexed by } \alpha \in \mathbb{R}.$$

- If Ψ is a FUNTF then $\alpha = 0$ designates the canonical dual, i.e.,

$$c_{m,n}^{0} = \frac{d}{s} \langle y_m, \Psi_n \rangle.$$

Frame coefficient images (continued)

- For each $m \in \{1, \dots, N\}$ choose an ℓ^1 sparse decomposition

$$y_m = \sum_{n=1}^{s} c_{m,n}^{\alpha(m)} \Psi_n$$

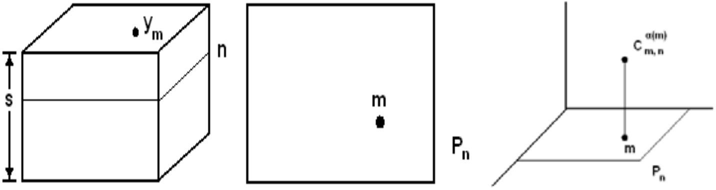
  defined by the inequality,

$$\forall\, \alpha, \quad \sum_{n=1}^{s} |c_{m,n}^{\alpha(m)}| \leq \sum_{n=1}^{s} |c_{m,n}^{\alpha}|.$$

- There is ℓ^0 theory.

Frame coefficient images (continued)

- Choose $n \in \{1, ..., s\}$. Take a slice, P_n, of the data cube at n. P_n contains N points m.



(a) Data Cube (b) Top Down Slice (c) $c_{m,n}^{\alpha(m)}$ defined

- The image with N pixels m, associated to the the frame element Ψ_n, is defined by $\{c_{m,n}^{\alpha(m)} \mid m = 1, ..., N\}$.

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Hyperspectral image processing

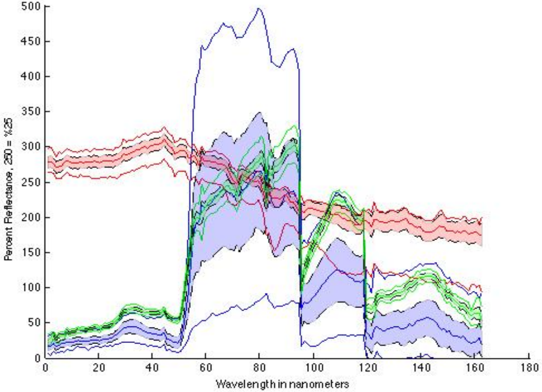
Urban data set classes

Figure: HYDICE Copperas Cove, TX — http://www.tec.army.mil/Hypercube/

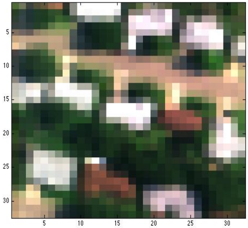
Urban data set classes

- NGA gave us 23 classes, associated with the different colors in the previous figure.
- In fact, if the 23 classes were to correspond roughly to orthogonal subspaces, then one cannot achieve effective dimension reduction less than dimension $d = 23$.
- However, we could have a frame with 23 elements in a space of reduced dimension $d < 23$.

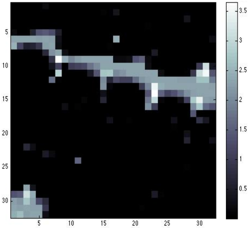
Spectral signatures of selected classes



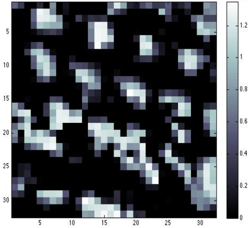
Frame coefficients



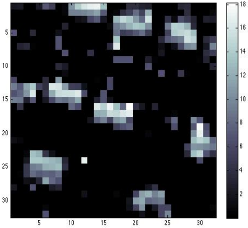
(a) Original



(b) Road coefficients

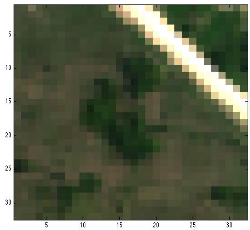


(c) Tree coefficients

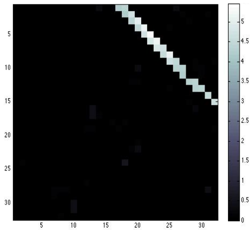


(d) White house coefficients

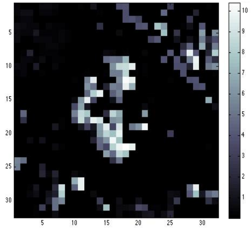
Frame coefficients



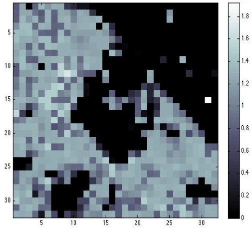
(a) Original



(b) Road coefficients

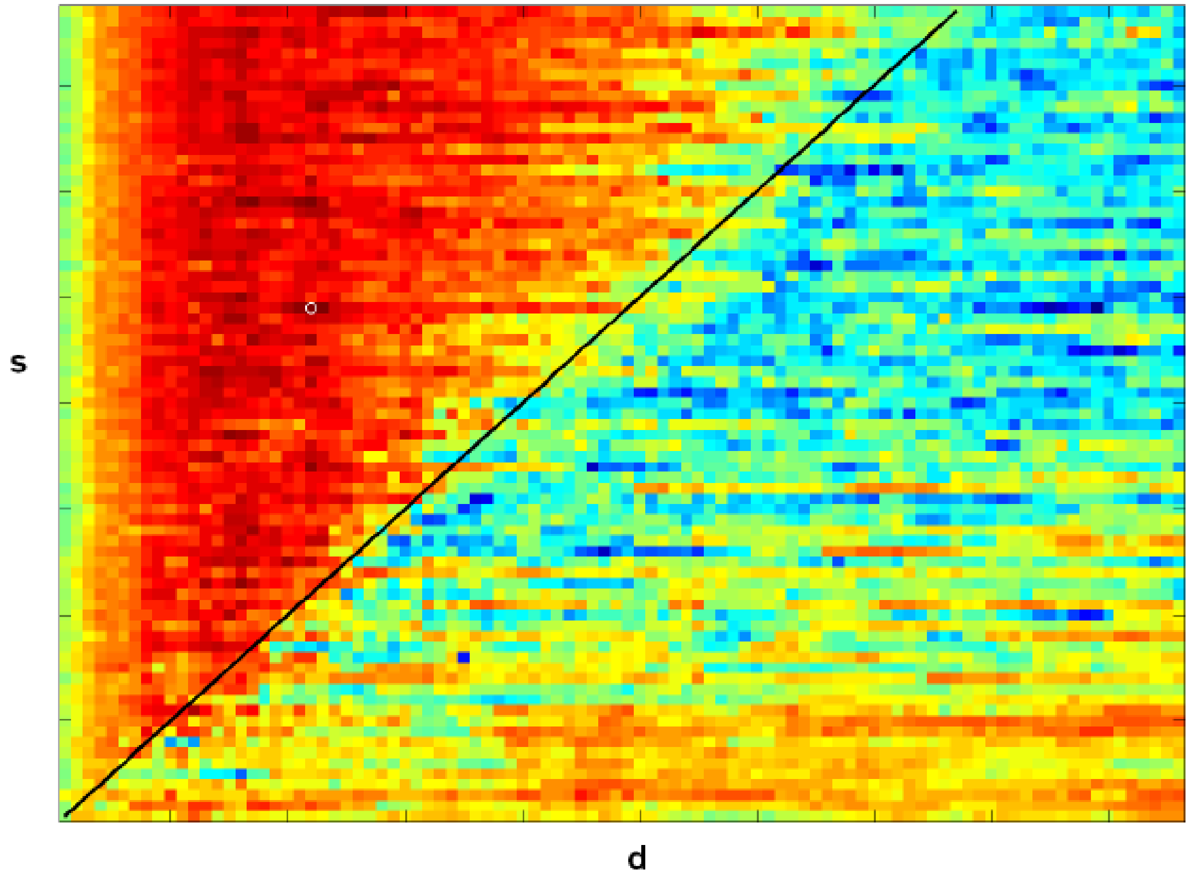


(c) Tree coefficients

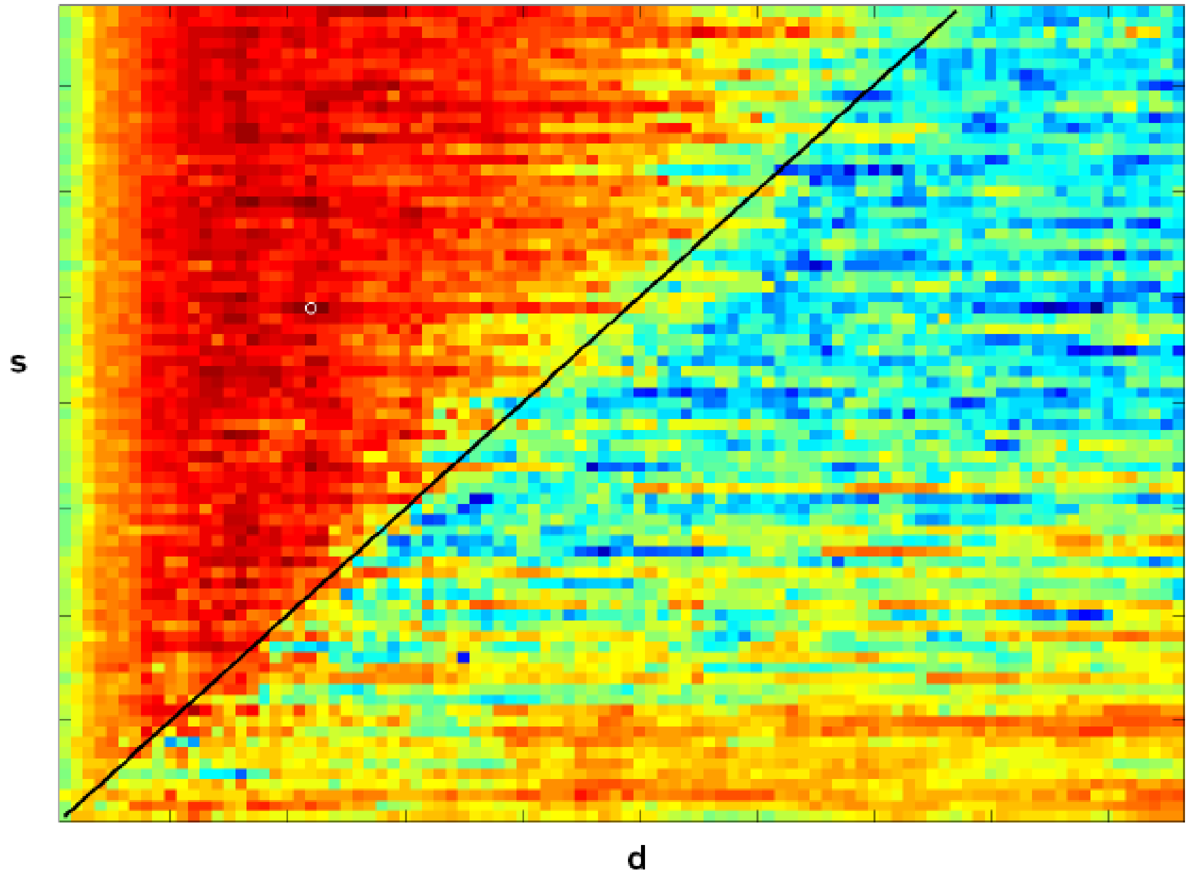


(d) Dirt/grass coefficients

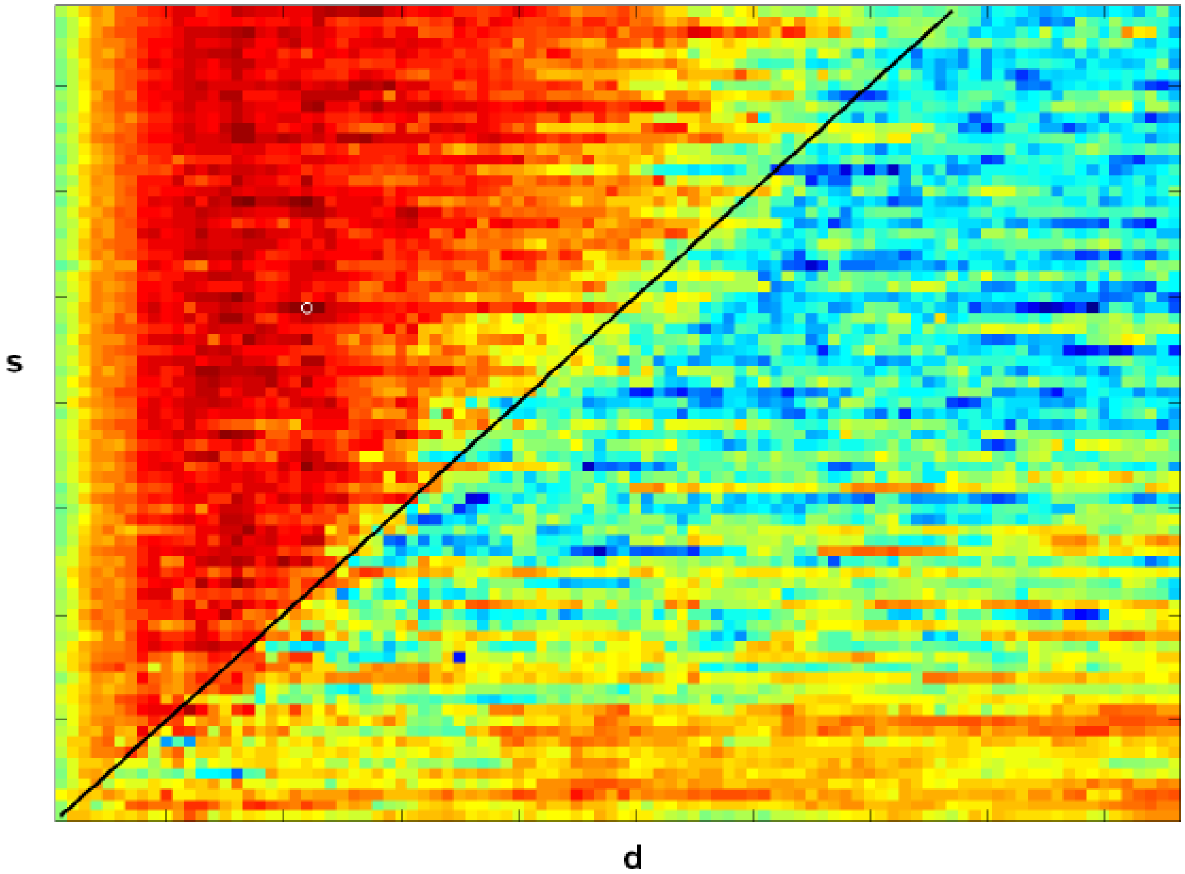
Overview of Classification Results



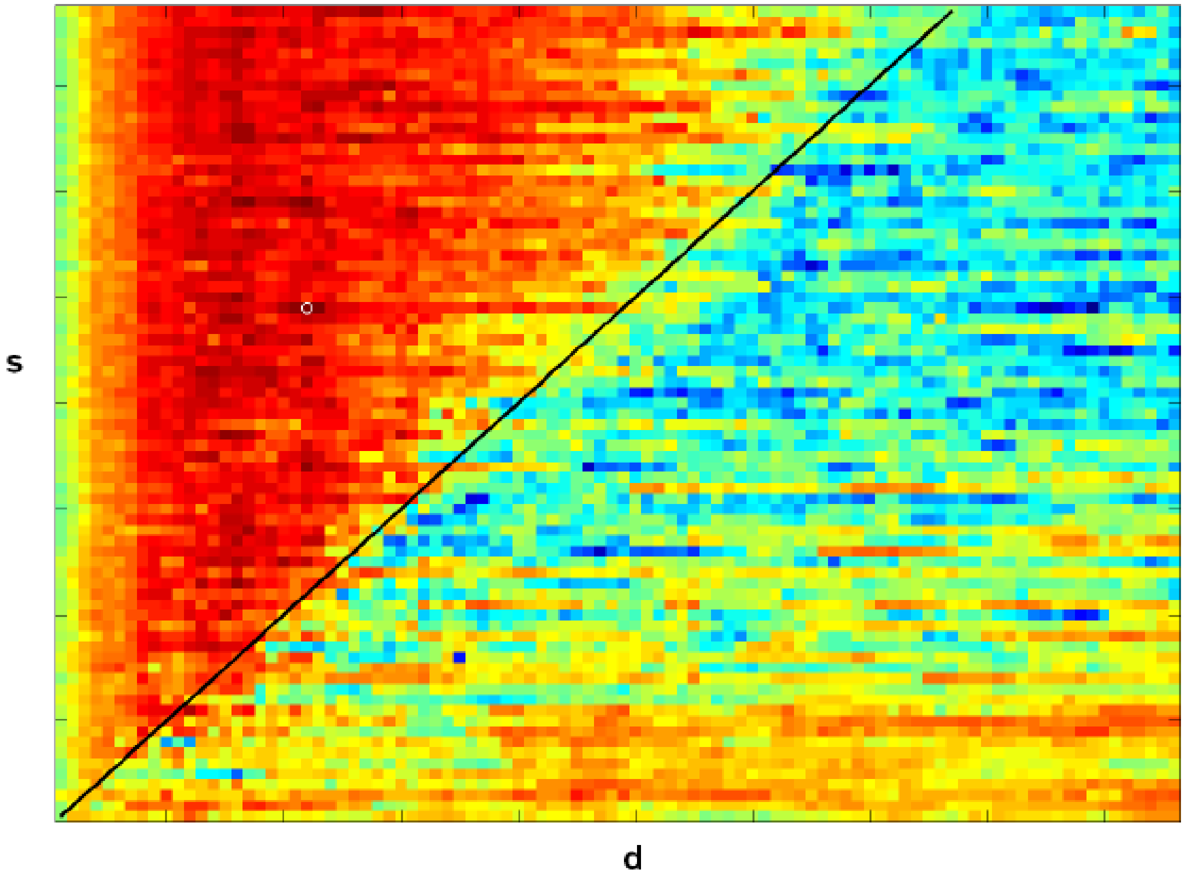
Overview of Classification Results



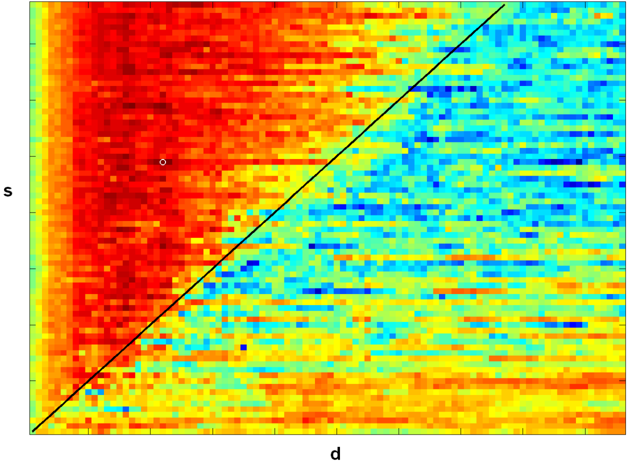
Overview of Classification Results



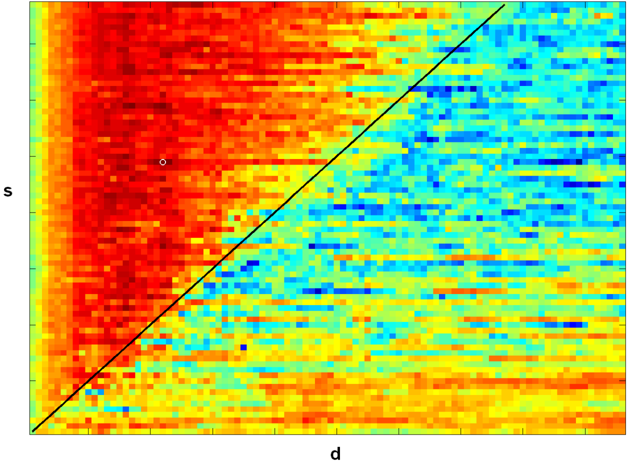
Overview of Classification Results



Overview of Classification Results



Overview of Classification Results



Outline

1. PCM - Sigma Delta Comparison
2. SRQP

A Quantization Problem

Given $x \in \mathbb{R}^d$, and a frame $\{e_k\}_{k=1}^N$ with Bessel map L, the quantization problem can be paraphrased as

$$\begin{array}{ll} \textit{minimize} & \|x - \frac{d}{N} L^* y\|, \\ \textit{subject to} & y \in \mathcal{C}_N := \{q \in \mathbb{R}^N : q_n = \pm 1\}. \end{array}$$

This problem is (NP) hard to solve. Instead, minimize a function like

$$F_x(y) = \lambda \|x - \frac{d}{N} L^* y\|^2 + P(y),$$

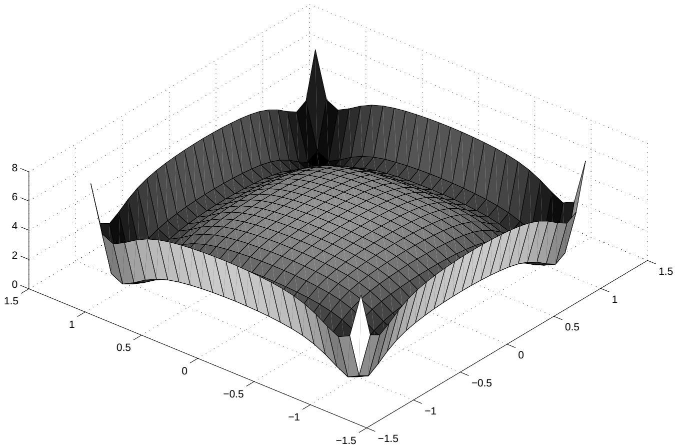
where $P : \mathbb{R}^N \to \mathbb{R}$, $P \geq 0$ is a penalty term, e.g.,

$$P(y) = \sum_{k=1}^N f(y_k), \quad f(t) = (1 - t^n)^2 + c(1 - t^m)$$

where $m \leq n$ are even positive integers, $c \geq 0$.

P is small on $\mathcal{C}_N$, gets larger away from $\mathcal{C}_N$.

A penalty function



$$P(y) = f(y_1) + f(y_2) \text{ with } n = 20,\ c = 1,\ m = 2.$$

Theorem 1

Let $c \geq 0$, let $m \leq n$ be two positive even integers, and let

$$F_x(y) = \lambda \|x - \frac{d}{N} L^* y\|_2^2 + \sum_{k=1}^{N} (1 - y_k^n)^2 + c(1 - y_k^m).$$

Every local minimum of F_x lies in the set

$$\mathcal{B}_{n,m,\lambda} := \left\{ y \in \mathbb{R}^N : \sum_{k=1}^{N} \left(y_k^n - \frac{1}{2} \right)^2 - \frac{cm}{2n} \sum_{k=1}^{N} y_k^m \leq \frac{N}{4} + \frac{\lambda}{4n} \|x\|^2 \right\}.$$

Theorem 1(Continued)

1. If y is a local minimum of F_x, then $\|y\|_\infty \leq R$, where R is the positive root of the polynomial

$$\Pi(\rho) = \rho^{2n} - \rho^n - \frac{cmN}{2n}\rho^m - \frac{\lambda\|x\|^2}{4n} - \frac{N-1}{4}.$$

   $R = 1 + \mathcal{O}(n^{-1})$ as $n \to \infty$, with other parameters fixed.

2. If y is a local minimum of F_x, then $|y_k| \geq r$ for at least $N - d$ indices, where r is the positive root of the polynomial

$$\pi(\rho) = \rho^{n-m}((2n-1)\rho^n - (n-1)) - \frac{cm(m-1)}{2n}.$$

   $r = 1 - \mathcal{O}(n^{-1})$ as $n \to \infty$, with other parameters fixed.

Theorem 2

Let y be a local minimizer of F_x, let $q = (round(y_1), \ldots, round(y_N))$, and let $J \subseteq \{1, \ldots, N\}$ be the set of indices where $|y_k| < r$. Then, we obtain the decomposition

$$x - \frac{d}{N} L^* q = x_\lambda + x_{ns} + x_J$$

where

$$\begin{aligned}
x_\lambda &= x - \frac{d}{N} L^* y, \quad and \quad \|x_\lambda\| = \mathcal{O}(\lambda^{-1}) \quad as \quad \lambda \to \infty, \\
x_{ns} &= \frac{d}{N} \sum_{k \notin J} (y_k - q_k) e_k, \quad and \quad \|x_{ns}\| = \mathcal{O}(n^{-1}) \quad as \quad n \to \infty, \\
x_J &= \frac{d}{N} \sum_{k \in J} (y_k - q_k) e_k, \quad where \quad |J| \leq d.
\end{aligned}$$

Some examples

Given $x \in \mathbb{R}^d$ and a frame given in the rows of the matrix L, the following short MATLAB script finds a local minimum of F_x.

```
n=100; c=1; m=4; [N d]=size(L);
lambda=2^(N/d); y0=Lx;
optns=optimset('Display','on','Largescale','on',...
   'MaxIter',1e+4, 'MaxFunEvals',1e+5,...
   'Gradobj','on','Hessian','on');

y=fminunc(@(y)F_x(y,x,L,lambda,n,c,m),y0,optns);
q=round(y);
```

Example 1

$d = 16$, $N = 216$, $\lambda = 2^{N/d}$, $n = 100$, $c = 1$, $m = 4$.

$$x = (-0.33778, 0.008157, 0.12914, 0.53439, 0.55974, -0.031804, 0.60443, -0.057976, -0.59448, 0.159230, 0.333, 0.35353, 0.88502, 0.5403, 0.47481, 0.73252),$$

Frame: Real harmonic frame H_N^d.

$$\begin{aligned} \|x_\lambda\| &= \|x - (d/N)L^*y\| = \texttt{4.1062e-005} \\ \|x_{ns}\| &= \texttt{0.0012903} \\ \|x_J\| &= \texttt{0.085473} \end{aligned}$$

$$card\{k : q_k = 0\} = 4,$$

$$\|x - (d/N)L^*q\| = 0.085565$$

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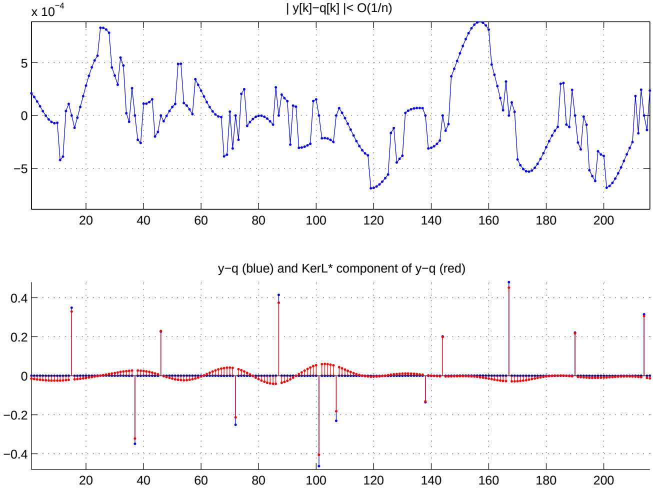


$$J = \{15, 37, 46, 72, 87, 101, 107, 138, 144, 167, 190, 214\}$$

Mean squared error

Example 2

$d = 2$, $N = 7$, $\lambda = 2^{N/d}$, $n = 100$, $c = 1$, $m = 4$.

Frame: 7th roots of unity frame for $\mathbb{R}^2$.

We quantized each point in the grid

$$\{(-1 + 0.02k, -1 + 0.02m) : k, m = 0, \dots, 100\}$$

Mean squared error

Example 3

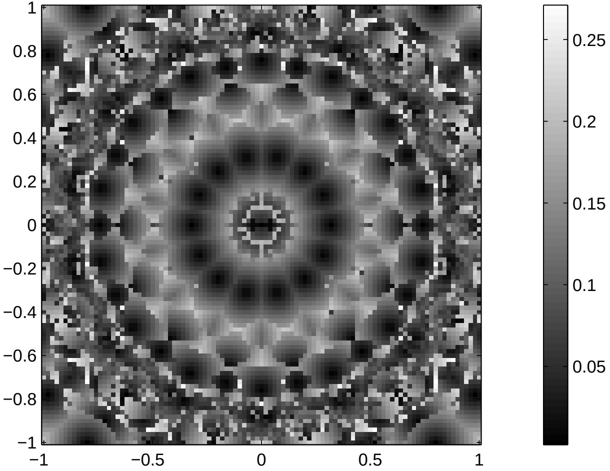
$d = 2$, $N = 10$, $\lambda = 2^{N/d}$, $n = 100$, $c = 1$, $m = 4$. The rows of L is a unit-norm tight frame for $\mathbb{R}^2$:

$$L = \begin{pmatrix} -0.26753 & -0.96355 \\ -0.25355 & -0.96732 \\ -0.67101 & -0.74145 \\ -0.81442 & -0.58028 \\ -0.97042 & -0.24142 \\ -0.99797 & 0.06367 \\ -0.8892 & 0.45752 \\ -0.73249 & 0.68078 \\ -0.64949 & 0.76037 \\ -0.25279 & 0.96752 \end{pmatrix}$$

We quantized each point in the grid

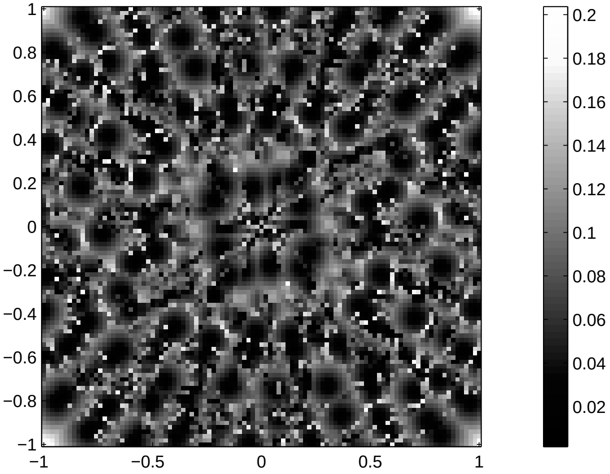
$$\{(-1 + 0.02k, -1 + 0.02m) : k, m = 0, \ldots, 100\}$$

Mean squared error

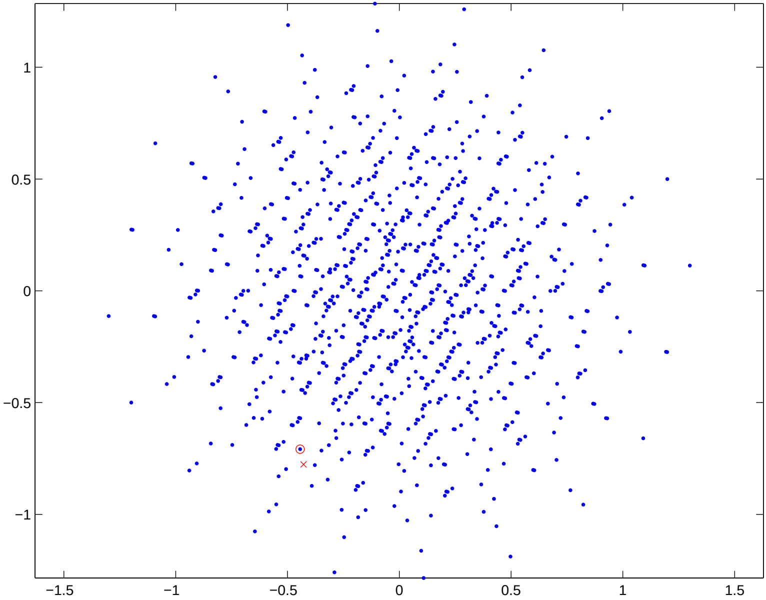


Average error = 0.1033,   Average error-squared = 0.0135

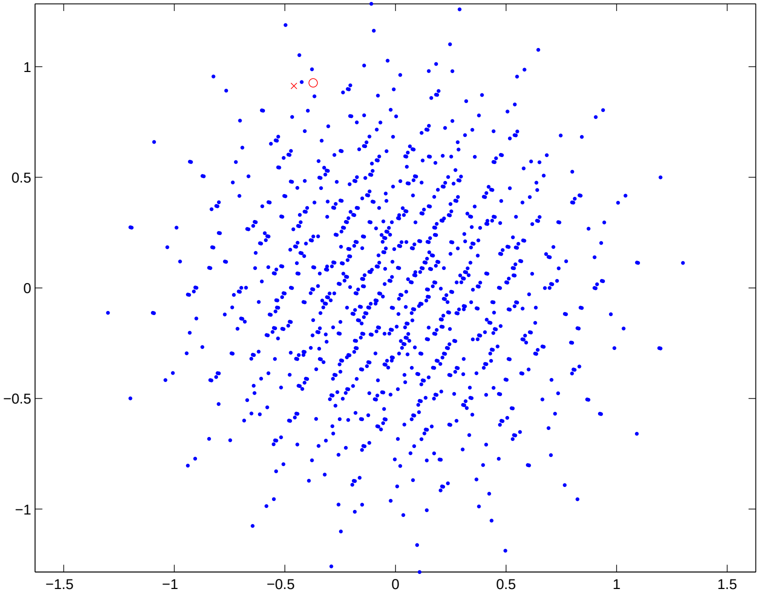
Mean squared error



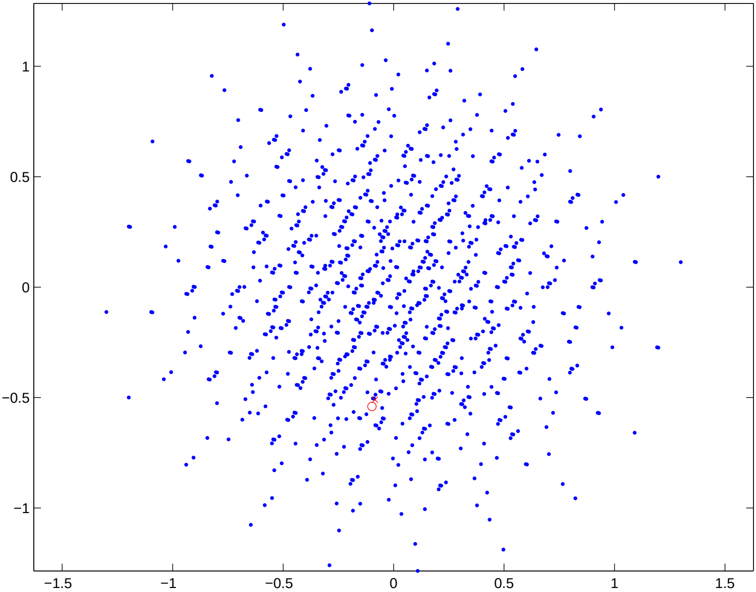
Average error = 0.0711, Average error-squared = 0.0064



o: Quantized estimate of x



o: Quantized estimate of x



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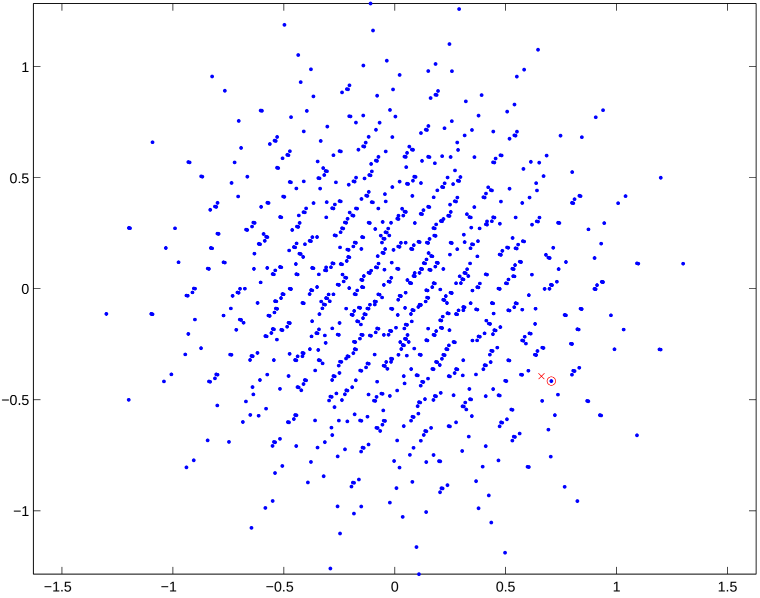
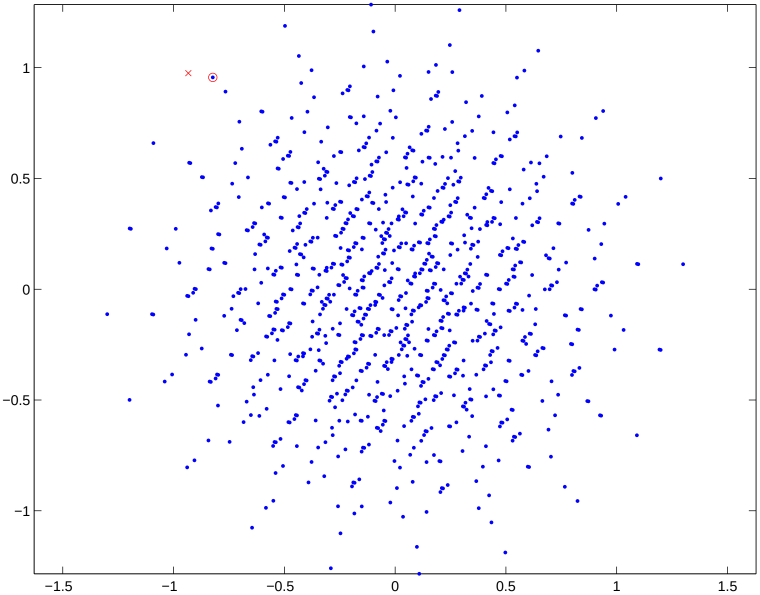
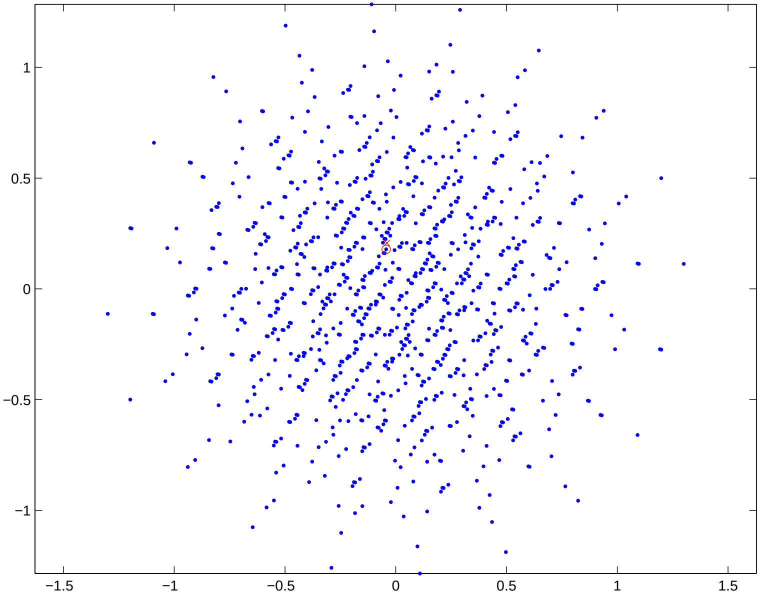
o: Quantized estimate of x



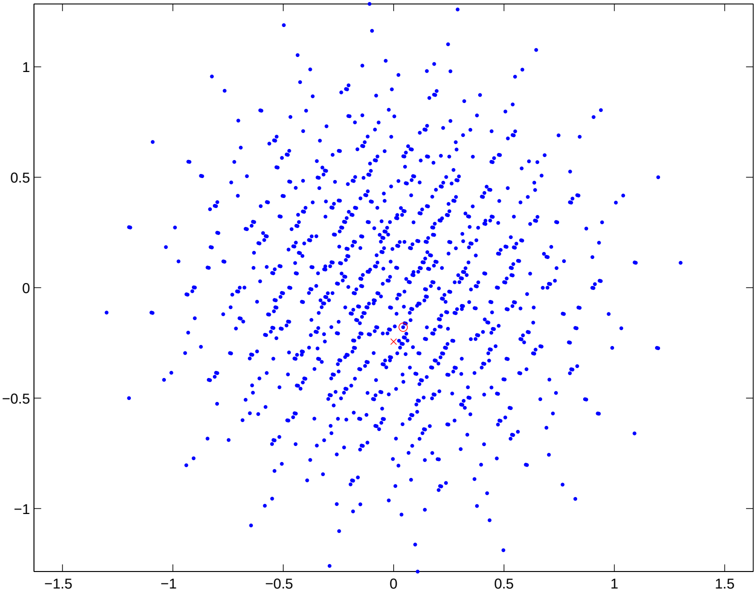
o: Quantized estimate of x



o: Quantized estimate of x



o: Quantized estimate of x



o: Quantized estimate of x

Shapiro-Rudin polynomials

Shapiro-Rudin polynomials

- The *Shapiro-Rudin polynomials*, $P_n(t)$, $Q_n(t)$, $n = 0, 1, 2, ...,$ are defined recursively in the following manner. For $t \in \mathbb{R}/\mathbb{Z}$,

$$\begin{aligned} P_0(t) = Q_0(t) &= 1, \\ P_{n+1}(t) &= P_n(t) + e^{2\pi i 2^n t} Q_n(t), \\ Q_{n+1}(t) &= P_n(t) - e^{2\pi i 2^n t} Q_n(t). \end{aligned}$$

- The number of terms in the n^{th} polynomial, $P_n(t)$ or $Q_n(t)$, is 2^n.
- Thus, the coefficients of each polynomial can be represented as a finite sequence of length 2^n of (± 1)s.

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Golay complementary pairs

- For any sequence $z = \{z_k\}_{k=0}^{n-1} \subseteq \mathbb{C}$ and for any $m \in \{0, 1, ..., n-1\}$, the m^{th} *aperiodic autocorrelation coefficient*, $A_z(m)$, is defined as

$$A_z(m) = \sum_{j=0}^{n-1-m} z_j \overline{z_{m+j}}.$$

- Two sequences, $p = \{p_k\}_{k=0}^{n-1}, q = \{q_k\}_{k=0}^{n-1} \subseteq \mathbb{C}$, are a *Golay complementary pair* if $A_p(0) + A_q(0) \neq 0$, and,

$$\forall m = 1, 2, ..., n-1, \qquad A_p(m) + A_q(m) = 0.$$

- For each n, the coefficients of P_n and Q_n, resp., are a Golay complementary pair.

Cusps

- A parametrized curve $\gamma : \mathbb{R} \to \mathbb{R}^2$, defined by $\gamma(t) = (u(t), v(t))$, has a *non-regular point* at $t = t_0$ if $\frac{du}{dt}|_{t=t_0} = \frac{dv}{dt}|_{t=t_0} = 0$. Otherwise, t_0 is a *regular point*.
- A non-regular point t_0 gives rise to a *quadratic cusp* for γ if $\left(\frac{d^2u}{dt^2}|_{t=t_0}, \frac{d^2v}{dt^2}|_{t=t_0}\right) \neq (0,0)$.
- A non-regular point t_0 gives rise to an *ordinary cusp* if it gives rise to a quadratic cusp, and $\left(\frac{d^2u}{dt^2}|_{t=t_0}, \frac{d^2v}{dt^2}|_{t=t_0}\right)$ and $\left(\frac{d^3u}{dt^3}|_{t=t_0}, \frac{d^3v}{dt^3}|_{t=t_0}\right)$ are linearly independent vectors of the real vector space $\mathbb{R}^2$.
- Let $P(z) = z^2 - 2z$ on $\mathbb{C}$, and define $\gamma(t) = P(e^{2\pi i t})$. Then, γ has a non-regular point at $t = t_0$ and gives rise to a quadratic cusp there.

Theorem

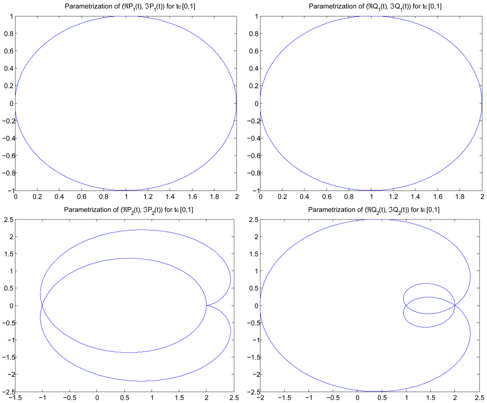
a. For each $n \in \mathbb{N}$, the parametrization $(Re\,(P_{2n}(t))\,, Im\,(P_{2n}(t)))$ gives rise to a quadratic cusp at $(2^n, 0)$, i.e., when $t = 0$.
b. Further, neither $(Re\,(P_{2n+1}(t))\,, Im\,(P_{2n+1}(t)))$ nor $(Re\,(Q_n(t))\,, Im\,(Q_n(t)))$ gives rise to a cusp when $t = 0$.

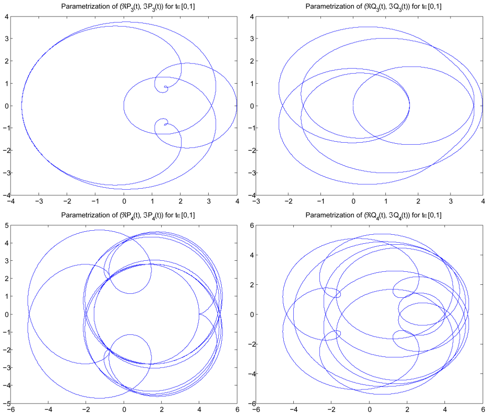
Remark

The Theorem does not contradict the fact that $P_{2n} : \mathbb{R} \longrightarrow \mathbb{C}$ is infinitely differentiable as a 1-periodic polynomial on $\mathbb{R}$.

Graphs of $P_n(t)$ and $Q_n(t)$ for n=1,2,3,4

Graphical parametrizations of $P_n(t)$ and $Q_n(t)$ by means of $(Re\,(P_n(t))\,,Im\,(P_n(t)))$ and $(Re\,(Q_n(t))\,,Im\,(Q_n(t)))$ for $n = 1, 2, 3, 4$.




Parametrization of $(\Re P_3(t), \Im P_3(t))$ for $t \in [0,1]$
Parametrization of $(\Re Q_3(t), \Im Q_3(t))$ for $t \in [0,1]$
Parametrization of $(\Re P_4(t), \Im P_4(t))$ for $t \in [0,1]$
Parametrization of $(\Re Q_4(t), \Im Q_4(t))$ for $t \in [0,1]$

A vector-valued ambiguity function

Outline

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- Originally, our problem was to construct libraries of phase-coded waveforms v parameterized by design variables, for communications and radar.
- A goal was to achieve diverse ambiguity function behavior of v by defining new classes of quadratic phase and number theoretic perfect autocorrelation codes u with which to define v.
- A realistic more general problem was to construct vector-valued waveforms v in terms of vector-valued perfect autocorrelation codes u. Such codes are relevant in light of vector sensor and MIMO capabilities and modeling.
- Example: Discrete time data vector $u(k)$ for a d-element array,

$$k \longmapsto u(k) = (u_0(k), \ldots, u_{d-1}(k)) \in \mathbb{C}^d.$$

We can have $\mathbb{R}^N \to GL(d, \mathbb{C})$, or even more general.

General problem and STFT theme

- Establish the theory of vector-valued ambiguity functions to estimate v in terms of ambiguity data.
- First, establish this estimation theory by defining the discrete periodic vector-valued ambiguity function in a natural way.
- Mathematically, this natural way is to formulate the discrete periodic vector-valued ambiguity function in terms of the Short Time Fourier Transform (STFT).

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STFT and ambiguity function

Short time Fourier transform – STFT

- The narrow band cross-correlation ambiguity function of v, w defined on $\mathbb{R}$ is

$$A(v,w)(t,\gamma) = \int_{\mathbb{R}} v(s+t)\overline{w(s)}e^{-2\pi i s\gamma}\,ds.$$

- $A(v,w)$ is the STFT of v with window w.
- The *narrow band radar ambiguity function* $A(v)$ of v on $\mathbb{R}$ is

$$A(v)(t,\gamma) = \int_{\mathbb{R}} v(s+t)\overline{v(s)}e^{-2\pi i s\gamma}\,ds$$

$$= e^{\pi i t\gamma}\int_{\mathbb{R}} v\left(s+\frac{t}{2}\right)\overline{v\left(s-\frac{t}{2}\right)}e^{-2\pi i s\gamma}\,ds, \text{ for } (t,\gamma)\in\mathbb{R}^2.$$

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Goal

- Let v be a phase coded waveform with N lags defined by the code u.
- Let u be N-periodic, and so $u : \mathbb{Z}_N \longrightarrow \mathbb{C}$, where $\mathbb{Z}_N$ is the additive group of integers modulo N.
- The *discrete periodic ambiguity function* $A_p(u) : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}$ is

$$A_p(u)(m, n) = \frac{1}{N} \sum_{k=0}^{N-1} u(m+k)\overline{u(k)}e^{-2\pi ikn/N}.$$

Goal

Given a vector valued N-periodic code $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$, construct the following in a meaningful, computable way:

- Generalized $\mathbb{C}$-valued periodic ambiguity function $A_p^1(u) : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}$
- $\mathbb{C}^d$-valued periodic ambiguity function $A_p^d(u) : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}^d$

The STFT is the *guide* and the *theory of frames* is the technology to obtain the goal.

Outline

Multiplication problem

- Given $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$.
- If $d = 1$ and $e_n = e^{2\pi in/N}$, then

$$A_p(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\langle u(m+k), u(k)e_{nk}\rangle.$$

Multiplication problem

To characterize sequences $\{E_k\} \subseteq \mathbb{C}^d$ and multiplications $*$ so that

$$A_p^1(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\langle u(m+k), u(k) * E_{nk}\rangle \in \mathbb{C}$$

is a meaningful and well-defined *ambiguity function.* This formula is clearly motivated by the STFT.

Ambiguity function assumptions

There is a natural way to address the multiplication problem motivated by the fact that $e_m e_n = e_{m+n}$. To this end, we shall make the *ambiguity function assumptions*:

- $\exists \, \{E_k\}_{k=0}^{N-1} \subseteq \mathbb{C}^d$ and a multiplication $*$ such that $E_m * E_n = E_{m+n}$ for $m, n \in \mathbb{Z}_N$;
- $\{E_k\}_{k=0}^{N-1} \subseteq \mathbb{C}^d$ is a tight frame for $\mathbb{C}^d$;
- $* : \mathbb{C}^d \times \mathbb{C}^d \longrightarrow \mathbb{C}^d$ is bilinear, in particular,

$$\left(\sum_{j=0}^{N-1} c_j E_j \right) * \left(\sum_{k=0}^{N-1} d_k E_k \right) = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} c_j d_k E_j * E_k .$$

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Calculation

- Let $\{E_j\}_j^{N-1} \subseteq \mathbb{C}^d$ satisfy the three ambiguity function assumptions.
- Given $u, v : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$ and $m, n \in \mathbb{Z}_N$.
- Then, one calculates

$$u(m) * v(n) \quad = \quad \frac{d^2}{N^2} \sum_{j=0}^{N-1} \sum_{s=0}^{N-1} \langle u(m), E_j \rangle \langle v(n), E_s \rangle E_{j+s}.$$

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Outline

$A^1_p(u)$ for DFT frames

- Let $\{E_j\}_j^{N-1} \subseteq \mathbb{C}^d$ satisfy the three ambiguity function assumptions.
- Further, assume that $\{E_j\}_{j=0}^{N-1}$ is a DFT frame, and let r designate a fixed column.
- Without loss of generality, choose the first d columns of the $N \times N$ DFT matrix.
- Then, one calculates

$$\begin{aligned} E_m * E_n(r) &= \frac{d^2}{N^2} \sum_{j=0}^{N-1} \sum_{s=0}^{N-1} \langle E_m, E_j \rangle \langle E_n, E_s \rangle E_{j+s}(r). \\ &= \frac{e_{(m+n)r}}{\sqrt{d}} = E_{m+n}(r). \end{aligned}$$

$A^1_p(u)$ for DFT frames

- Thus, for DFT frames, $*$ is componentwise multiplication in $\mathbb{C}^d$ with a factor of $\sqrt{d}$.
- In this case $A^1_p(u)$ is well-defined for $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$ by

$$\begin{aligned} A^1_p(u)(m,n) &= \frac{1}{N}\sum_{k=0}^{N-1}\langle u(m+k), u(k) * E_{nk}\rangle \\ &= \frac{d}{N^2}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\langle E_j, u(k)\rangle\langle u(m+k), E_{j+nk}\rangle. \end{aligned}$$

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Remark

- In the previous DFT example, $*$ is intrinsically related to the "addition" defined on the indices of the frame elements, viz., $E_m * E_n = E_{m+n}$.
- Alternatively, we could have $E_m * E_n = E_{m \bullet n}$ for some function $\bullet : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{Z}_N$, and, thereby, we could use frames which are not FUNTFs.
- Given a bilinear multiplication $* : \mathbb{C}^d \times \mathbb{C}^d \longrightarrow \mathbb{C}^d$, we can find a frame $\{E_j\}_j$ and an index operation $\bullet$ with the $E_m * E_n = E_{m \bullet n}$ property.
- If $\bullet$ is the multiplication for a group, possibly non-abelian and/or infinite, we may reverse the process and find a FUNTF and bilinear multiplication $*$ with the $E_m * E_n = E_{m \bullet n}$ property.

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$A_p^1(u)$ for cross product frames

- Take $* : \mathbb{C}^3 \times \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ to be the cross product on $\mathbb{C}^3$ and let $\{i, j, k\}$ be the standard basis.
- $i * j = k$, $j * i = -k$, $k * i = j$, $i * k = -j$, $j * k = i$, $k * j = -i$, $i * i = j * j = k * k = 0$. $\{0, i, j, k, -i, -j, -k, \}$ is a tight frame for $\mathbb{C}^3$ with frame constant 2. Let $E_0 = 0, E_1 = i, E_2 = j, E_3 = k, E_4 = -i, E_5 = -j, E_6 = -k$.
- The index operation corresponding to the frame multiplication is the non-abelian operation $\bullet : \mathbb{Z}_7 \times \mathbb{Z}_7 \longrightarrow \mathbb{Z}_7$, where $1 \bullet 2 = 3, 2 \bullet 1 = 6, 3 \bullet 1 = 2, 1 \bullet 3 = 5, 2 \bullet 3 = 1, 3 \bullet 2 = 4, 1 \bullet 1 = 2 \bullet 2 = 3 \bullet 3 = 0, n \bullet 0 = 0 \bullet n = 0, 1 \bullet 4 = 0, 1 \bullet 5 = 6, 1 \bullet 6 = 2, 4 \bullet 1 = 0, 5 \bullet 1 = 3, 6 \bullet 1 = 5, 2 \bullet 4 = 3, 2 \bullet 5 = 0$, etc.
- The three ambiguity function assumptions are valid and so we can write the cross product as

$$u \times v = u * v = \frac{1}{2^2} \sum_{s=1}^{6} \sum_{t=1}^{6} \langle u, E_s \rangle \langle v, E_t \rangle E_{s \bullet t}.$$

- Consequently, $A_p^1(u)$ can be well-defined.

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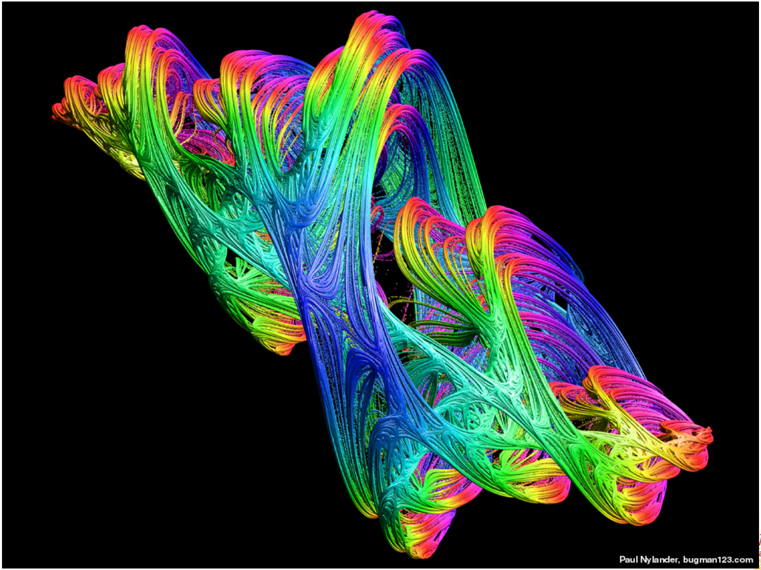
Inverse 4D Quaternion Julia Set



Vector-valued ambiguity function $A_p^d(u)$

- Let $\{E_j\}_j^{N-1} \subseteq \mathbb{C}^d$ satisfy the three ambiguity function assumptions.
- Given $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$.
- The following definition is clearly *motivated* by the STFT.

Definition

$A_p^d(u) : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}^d$ is defined by

$$A_p^d(u)(m,n) = \frac{1}{N} \sum_{k=0}^{N-1} u(m+k) * \overline{u(k)} * \overline{E_{nk}}.$$

Outline

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STFT formulation of $A_p(u)$

- The discrete periodic ambiguity function of $u : \mathbb{Z}_N \longrightarrow \mathbb{C}$ can be written as

$$A_p(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1} \langle \tau_m u(k), F^{-1}(\tau_n \hat{u})(k)\rangle,$$

  where $\tau_(m)u(k) = u(m+k)$ is translation by m and $F^{-1}(u)(k)) = \check{u}(k)$ is Fourier inversion.
- As such we see that $A_p(u)$ has the form of a STFT.
- We shall develop a vector-valued DFT theory to *verify* (not just *motivate*) that $A_p^d(u)$ is an STFT in the case $\{E_k\}_{k=0}^{N-1}$ is a DFT frame for $\mathbb{C}^d$.

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DFT frames and the vector-valued DFT

Definition

Given $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$, and let $\{E_k\}_{k=0}^{N-1}$ be a DFT frame for $\mathbb{C}^d$. The *vector-valued discrete Fourier transform* of u is

$$\forall n \in \mathbb{Z}_N,\ F(u)(n) = \hat{u}(n) = \sum_{m=0}^{N-1} u(m) * E_{mn},$$

where $*$ is pointwise (coordinatewise) multiplication.

- The vector-valued DFT inversion formula is valid if N is prime.
- Vector-valued DFT uncertainty principle inequalities are valid, similar to Tao-Candes in compressive sensing.

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Vector-valued Fourier inversion theorem

- Inversion process for the vector-valued case is analogous to the 1-dimensional case.
- We must define a new multiplication in the frequency domain to avoid divisibility problems.
- Define the weighted multiplication $(*) : \mathbb{C}^d \times \mathbb{C}^d \longrightarrow \mathbb{C}^d$ by $u(*)v = u * v * \omega$ where $\omega = (\omega_1, \ldots, \omega_d)$ has the property that each $\omega_n = \frac{1}{\#\{m \in \mathbb{Z}_N : mn = 0\}}$.
- For the following theorem assume $d << N$ or N prime.

Theorem - Vector-valued Fourier inversion

The vector valued Fourier transform F is an isomorphism from $\ell^2(\mathbb{Z}_N)$ to $\ell^2(\mathbb{Z}_N, \omega)$ with inverse

$$\forall\ m \in \mathbb{Z}_N, \quad F^{-1}(m) = u(m) = \frac{d}{N} \sum_{n=0}^{N-1} \hat{u}(n) * E_{-mn} * \omega.$$

N prime implies F is unitary.

$A_p^d(u)$ as an STFT

- Given $u, v : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$, and let $\{E_k\}_{k=0}^{N-1}$ be a DFT frame for $\mathbb{C}^d$.
- $u * \overline{v}$ denotes pointwise (coordinatewise) multiplication with a factor of $\sqrt{d}$.
- We compute

$$A_p^d(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}(\tau_m u(k)) * \overline{F^{-1}(\tau_n \hat{u})(k)}.$$

- Thus, $A_p^d(u)$ is compatible with point of view of defining a vector-valued ambiguity function in the context of the STFT.

Outline

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Epilogue

- If $(G, \bullet)$ is a finite group with representation $\rho : G \longrightarrow GL(\mathbb{C}^d)$, then there is a frame $\{E_n\}_{n \in G}$ and bilinear multiplication, $* : \mathbb{C}^d \times \mathbb{C}^d \longrightarrow \mathbb{C}^d$, such that $E_m * E_n = E_{m \bullet n}$. Thus, we can develop $A_p^d(u)$ theory in this setting.
- Analyze ambiguity function behavior for (phase-coded) vector-valued waveforms $v : \mathbb{R} \longrightarrow \mathbb{C}^d$, defined by $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$ as

$$v = \sum_{k=0}^{N-1} u(k) \mathbb{1}_{[kT, (k+1)T)},$$

in terms of $A_p^d(u)$. (See Figure)

Computation of $u : \mathbb{Z}_N \to \mathbb{C}^d$ from ambiguity

- CAZAC and waveform computation of $u : \mathbb{Z}_N \to \mathbb{C}^d$ from A(u): Let A_u be the $N \times N$ matix, $(A(u)(m, n))$. Define the $N \times N$ matrix $U = (U_{i,j})$, where $U_{i,j} = \langle u(i + j), u(j)\rangle$. Then

$$U = A_u D_N, \quad \text{where} \quad D_N = DFT \text{ matrix.}$$

- Let $d = 1$. Note that $U_{k,0} = u(k)\overline{u(0)}$. Hence, if we know the values of the ambiguity function, and, thus, the ambiguity function matrix A_u, then the sequence u, which generates it, can be computed as long as $u(0) \neq 0$. In fact, if $u(0) = 1$ then $u(k) = (A_u D_N)(k, 0)$.
- Similar result for $A_V(u)$ using our vector-valued Fourier analysis.
- Now we can address the classical *radar ambiguity problem*: Find the structure of all $z : \mathbb{Z}_N \to \mathbb{C}^d$ for which $|A(u)| = |A(z)|$ on $X \subseteq \mathbb{Z}_N \times \mathbb{Z}_N$.

Computation of $u : \mathbb{Z}_N \to \mathbb{C}^d$ from ambiguity

- CAZAC and waveform computation of $u : \mathbb{Z}_N \to \mathbb{C}^d$ from A(u): Let A_u be the $N \times N$ matix, $(A(u)(m,n))$. Define the $N \times N$ matrix $U = (U_{i,j})$, where $U_{i,j} = \langle u(i+j), u(j)\rangle$. Then

$$U = A_u D_N, \quad \text{where} \quad D_N = DFT \text{ matrix.}$$

- Let $d = 1$. If $u(0) = 1$ then $u(k) = (A_u D_N)(k, 0)$.
- Similar result for $A_V(u)$ using our vector-valued Fourier analysis.
- We are addressing the classical *radar ambiguity problem*: Find the structure of all $z : \mathbb{Z}_N \to \mathbb{C}^d$ for which $|A(u)| = |A(z)|$ on $X \subseteq \mathbb{Z}_N \times \mathbb{Z}_N$. This is not even resolved for the narrow-band case.
- The radar ambiguity problem is closely related to our approach of achieving diverse ambiguity function behavior.

Thats all folks!

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Σ-Δ and analytic number theory

Even – odd

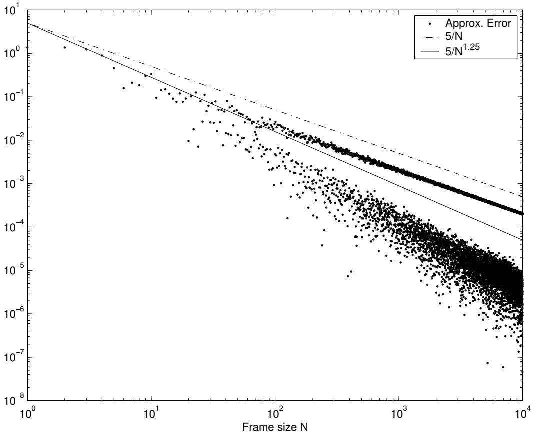

Figure: log-log plot of $\|x - \widetilde{x}_N\|$.

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Even – odd

$E_N = \{e_n^N\}_{n=1}^N$, $e_n^N = (\cos(2\pi n/N), \sin(2\pi n/N))$. Let $x = (\frac{1}{\pi}, \sqrt{\frac{3}{17}})$.

$$x = \frac{d}{N}\sum_{n=1}^{N} x_n^N e_n^N, \quad x_n^N = \langle x, e_n^N \rangle.$$

Let $\widetilde{x}_N$ be the approximation given by the 1st order $\Sigma\Delta$ quantizer with alphabet $\{-1, 1\}$ and natural ordering.

Improved estimates

$E_N = \{e_n^N\}_{n=1}^N$, Nth roots of unity FUNTFs for $\mathbb{R}^2$, $x \in \mathbb{R}^2$, $\|x\| \le (K - 1/2)\delta$.

$$\text{Quantize} \qquad x = \frac{d}{N}\sum_{n=1}^{N} x_n^N e_n^N, \quad x_n^N = \langle x, e_n^N\rangle$$

using 1st order $\Sigma\Delta$ scheme with alphabet $\mathcal{A}_K^\delta$.

Theorem

If N is even and large then $\|x - \widetilde{x}\| \le B_x \frac{\delta \log N}{N^{5/4}}$.

If N is odd and large then $A_x \frac{\delta}{N} \le \|x - \widetilde{x}\| \le B_x \frac{(2\pi+1)d}{N} \frac{\delta}{2}$.

- The proof uses a theorem of Güntürk (from complex or harmonic analysis); and Koksma and Erdös-Turán inequalities and van der Corput lemma (from analytic number theory).
- The Theorem is true for harmonic frames for $\mathbb{R}^d$.

Proof of Improved Estimates theorem

- If N is even and large then $\|x - \widetilde{x}\| \leq B_x \frac{\delta \log N}{N^{5/4}}$.
  If N is odd and large then $A_x \frac{\delta}{N} \leq \|x - \widetilde{x}\| \leq B_x \frac{(2\pi+1)d}{N} \frac{\delta}{2}$.
- $\forall N, \{e_n^N\}_{n=1}^N$ is a FUNTF.

$$
\begin{aligned}
x - \widetilde{x}_N &= \frac{d}{N}\left(\sum_{n=1}^{N-2} v_n^N (f_n^N - f_{n+1}^N) + v_{N-1}^N f_{N-1}^N + u_N^N e_N^N \right) \\
f_n^N &= e_n^N - e_{n+1}^N, \quad v_n^N = \sum_{j=1}^{n} u_j^N, \quad \widetilde{u}_n^N = \frac{u_n^N}{\delta}
\end{aligned}
$$

- $\boxed{\text{To bound } v_n^N.}$

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Koksma Inequality

Definition

The *discrepancy* D_N of a finite sequence $x_1, \ldots, x_N$ of real numbers is $D_N = D_N(x_1, \ldots, x_N) = \sup_{0 \leq \alpha \langle \beta \leq 1} \left| \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{[\alpha,\beta)}(\{x_n\}) - (\beta - \alpha) \right|$, where $\{x\} = x - \lfloor x \rfloor$.

Theorem(Koksma Inequality)

$g : [-1/2, 1/2) \to \mathbb{R}$ of bounded variation and $\{\omega_j\}_{j=1}^{n} \subset [-1/2, 1/2) \Longrightarrow$

$$\left| \frac{1}{n} \sum_{j=1}^{n} g(\omega_j) - \int_{-\frac{1}{2}}^{\frac{1}{2}} g(t)dt \right| \leq \mathrm{Var}(g)\mathrm{Disc}\left(\{\omega_j\}_{j=1}^{n} \right).$$

With $g(t) = t$ and $\omega_j = \widetilde{u}_j^N$, $\boxed{|v_n^N| \leq n\delta \mathrm{Disc}\left(\{\widetilde{u}_j^N\}_{j=1}^{n} \right).}$

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Erdös-Turán Inequality

$$\exists C > 0, \forall K, \mathrm{Disc}\left(\{\widetilde{u}_n^N\}_{n=1}^{j}\right) \leq C\left(\frac{1}{K} + \frac{1}{j}\sum_{k=1}^{K}\frac{1}{k}\left|\sum_{n=1}^{j} e^{2\pi i k \widetilde{u}_n^N}\right|\right).$$

To approximate the exponential sum.

Approximation of Exponential Sum

Güntürk's Proposition (1)

$\forall N, \exists X_N \in \mathcal{B}_{\Omega/N}$ such that , $\forall n = 0, \ldots, N$

$$X_N(n) = u_n^N + c_n \frac{\delta}{2}, \; c_n \in \mathbb{Z}$$

and, $\forall t$,

$$\left| X_N'(t) - h\left(\frac{t}{N}\right) \right| \leq B \frac{1}{N}$$

Bernstein's Inequality (2)

If $x \in \mathcal{B}_\Omega$, then $\|x^{(r)}\|_\infty \leq \Omega^r \|x\|_\infty$

Approximation of Exponential Sum

(1)+(2)

$$\forall t,\ \left|X_N''(t) - \frac{1}{N}h'\left(\frac{t}{N}\right)\right| \leq B\frac{1}{N^2}$$

- $\widehat{\mathcal{B}}_\Omega = \{T \in A'(\widehat{\mathbb{R}})\ :\ \operatorname{supp} T \subseteq [-\Omega, \Omega]\}$
- $\mathcal{M}_\Omega = \{h \in \mathcal{B}_\Omega\ :\ h' \in L^\infty(\mathbb{R})$ and all zeros of h' on $[0,1]$ are simple$\}$
- We assume
  $\exists h \in \mathcal{M}_\Omega$ such that $\forall N$ and $\forall\, 1 \leq n \leq N,\ h(n/N) = x_n^N$.

Van der Corput Lemma

- Let a, b be integers with $a < b$, and let f satisfy $f'' \geq \rho > 0$ on $[a, b]$ or $f'' \leq -\rho < 0$ on $[a, b]$. Then

$$\left|\sum_{n=a}^{b} e^{2\pi i f(n)}\right| \leq \Big(\big|f'(b) - f'(a)\big| + 2\Big)\Big(\frac{4}{\sqrt{\rho}} + 3\Big).$$

$$\Downarrow$$

- $\forall 0 < \alpha < 1, \exists N_\alpha$ such that $\forall N \geq N_\alpha$,

$$\left|\sum_{n=1}^{j} e^{2\pi i k \widetilde{u}_n^N}\right| \leq B_x N^\alpha + B_x \frac{\sqrt{k} N^{1-\frac{\alpha}{2}}}{\sqrt{\delta}} + B_x \frac{k}{\delta}.$$

Choosing appropriate α and K

Putting $\alpha = 3/4$, $K = N^{1/4}$ yields

$$\exists \widetilde{N} \text{ such that } \forall N \geq \widetilde{N}, \mathrm{Disc}\left(\{\widetilde{u}_n^N\}_{n=1}^j\right) \leq B_x \frac{1}{N^{\frac{1}{4}}} + B_x \frac{N^{\frac{3}{4}} \log(N)}{j}$$

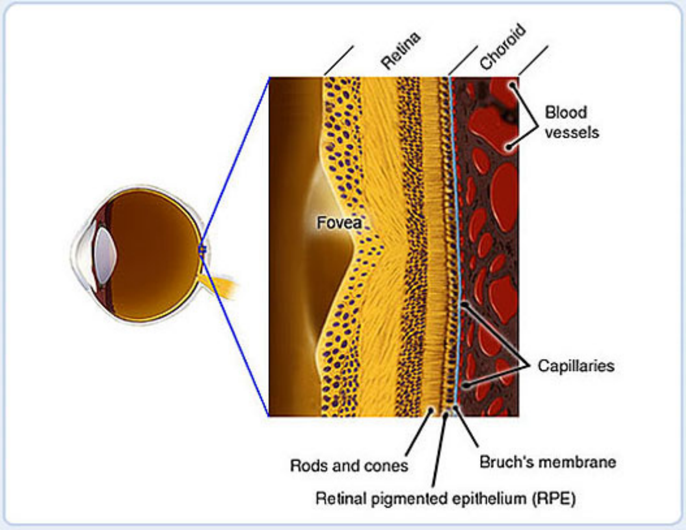
$$\Downarrow$$

Conclusion

$$\forall n = 1, \dots, N, \ |v_n^N| \leq B_x \delta N^{\frac{3}{4}} \log N$$

Retinal imaging and pre-drusen problem

Retinal Structure



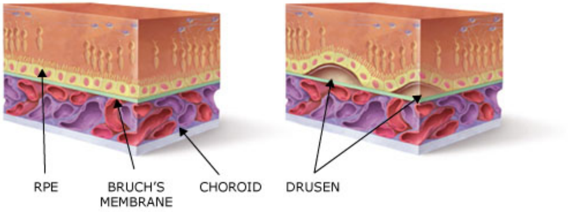
Age Related Macula Degeneration (AMD)

Serious Eye Desease

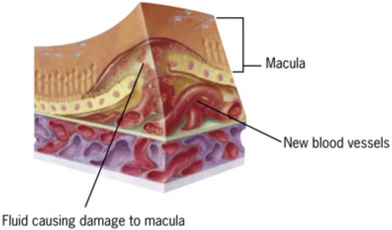
- Pathology:
  - irregular lipofuscine deposits, irregular vessel growth
  - losing the ability to focus (late stages: blindness)
- Relevance:
  - leading cause of blindness in elderly population
- Diagnostics:
  - imaging
  - visual tests
- Therapy:
  - currently no effective treatment

Way Out:

- detect/classify early stages → intervene → prevent progress



(a) Drusen: accumulated photoproducts



(b) Choroidal Neovascularization: wet AMD

Retinal imaging and pre-drusen problem

- The rods and cones of the eye receive light photons, and transform them into electric signals to the brain.
- These transformations leave lipofuscin deposits, leading to drusen in system malfunction, in the retinal pigmented epithelium (RPE) cells.
- The autofluorescent lipofuscin is comprised of several chemicals.
- Problem. Determine the existence of the composition of such chemicals, called *pre-drusen*, as an early indicator of AMD.

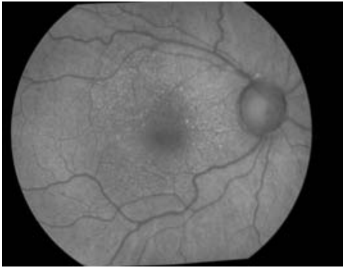
Multispectral retinal imaging

Image processing to detect pre-drusen

Retinal data

- Retinal data is collected using a Fundus camera.
- Light is shown into the eye at different wavelengths, and the fluorescence is measured.
- The images are then registered so that they align properly.

(a) Fundus camera

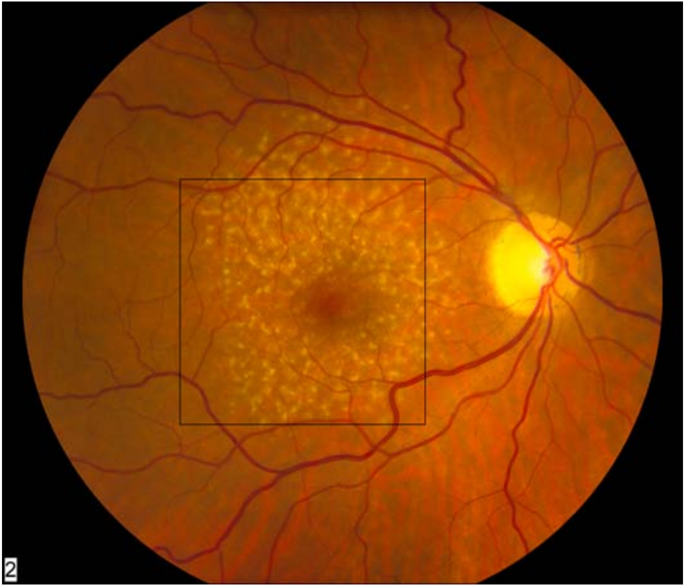


(b) Retinal fluorescence data

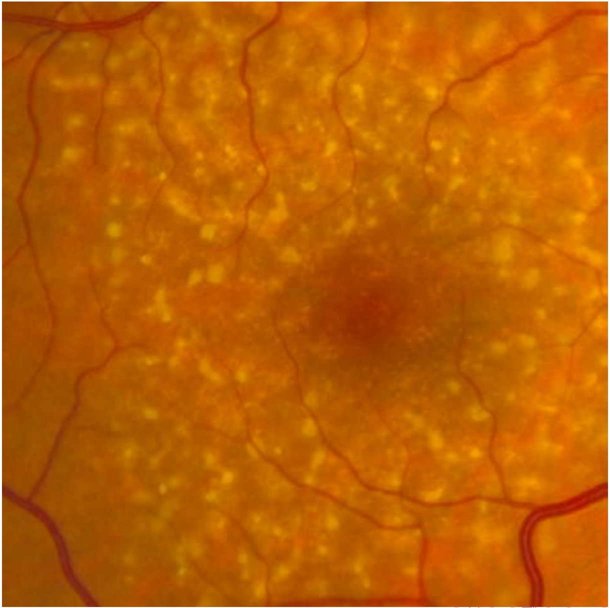
Multispectral retinal imaging data

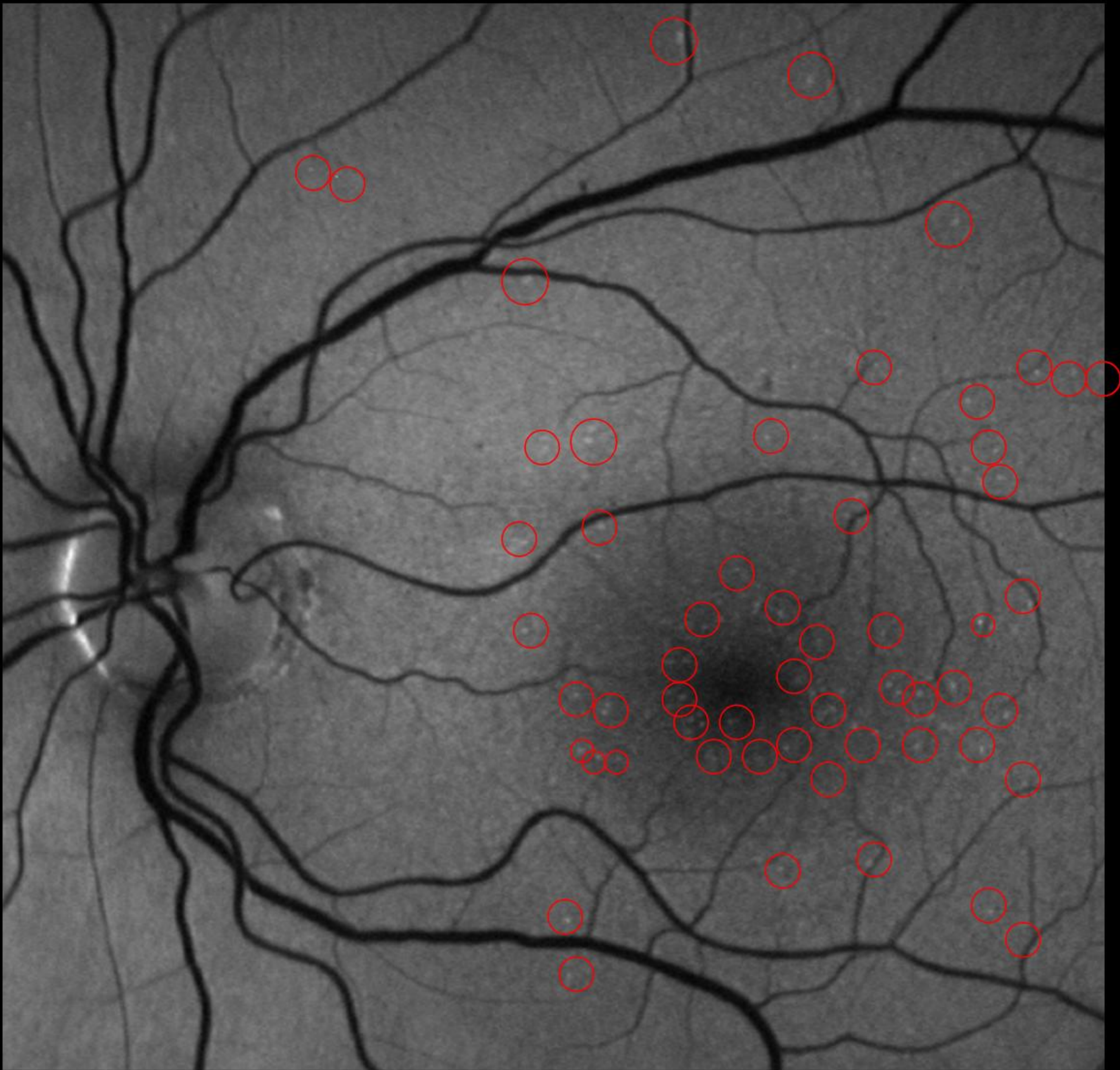
- A fundus camera is a low power microscope and camera to photograph the interior surface of the eye (retina, optic disc, macula, fovea, etc.), i.e., the fundus, opposite the lens.
- Aim. Obtain high resolution maps of macular pigment and retinal lipofuscin.
- Setup. Use 2 excitation bands and 4 absorption bands. Each of the 8 pairs of excitation/absorption filters produces an image.
- Data. Thus, our present multispectral imaging data cube consists of $10^3 \times 10^3$ pixels determined by 8 bands, i.e., 10^6 vectors in $\mathbb{R}^8$.

Color fundus image (shows Drusen)

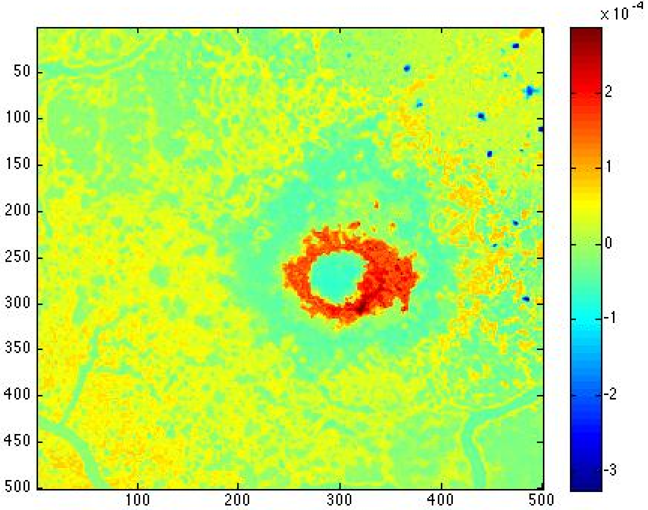


Enlarged color fundus image

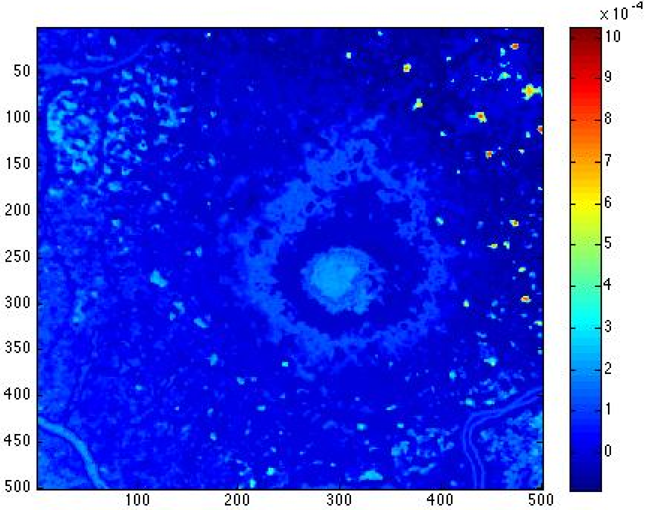




Pre-Drusen Images



Pre-Drusen Images



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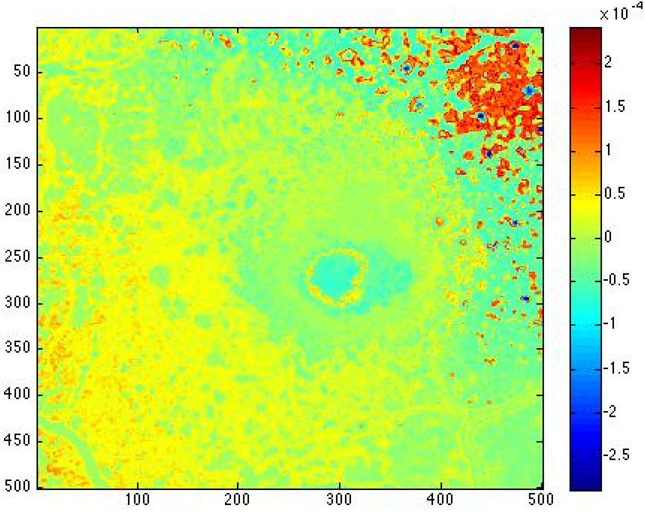
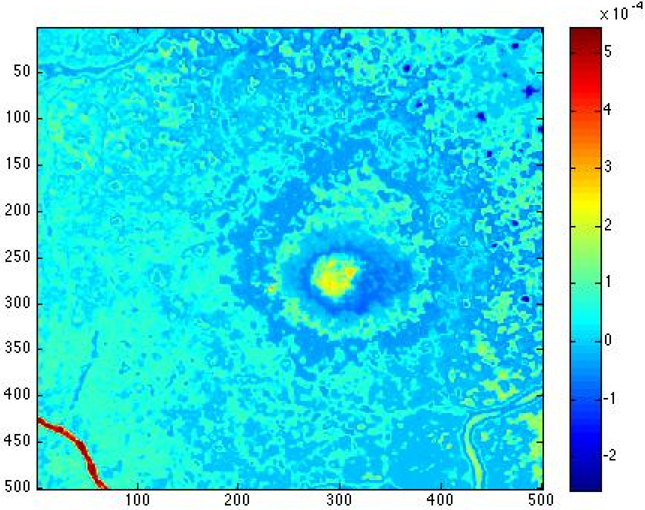


Pre-Drusen Images



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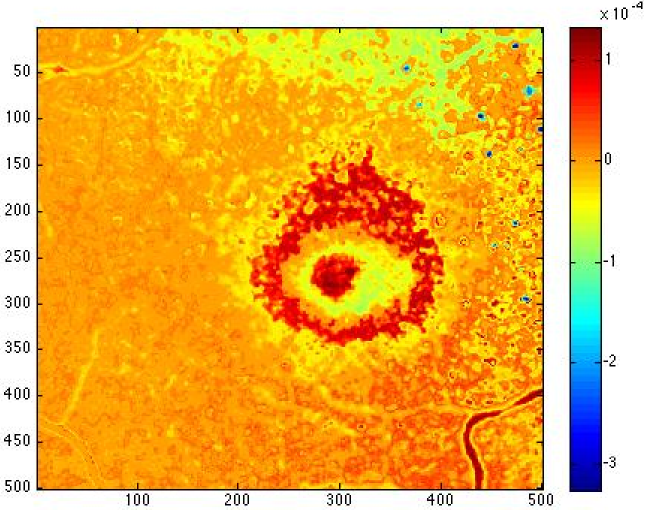

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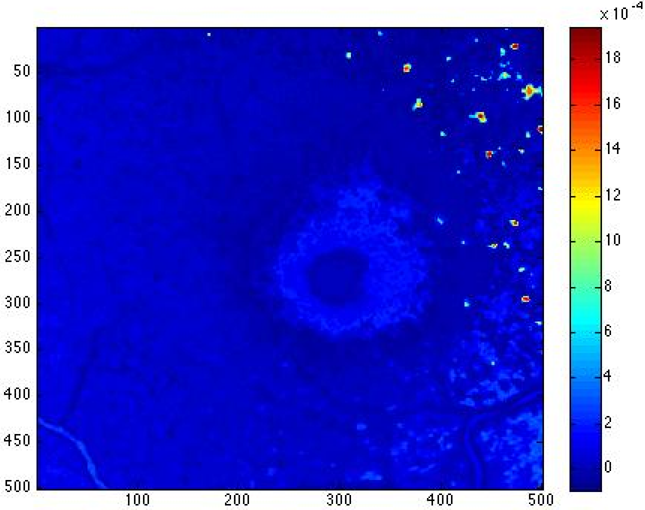
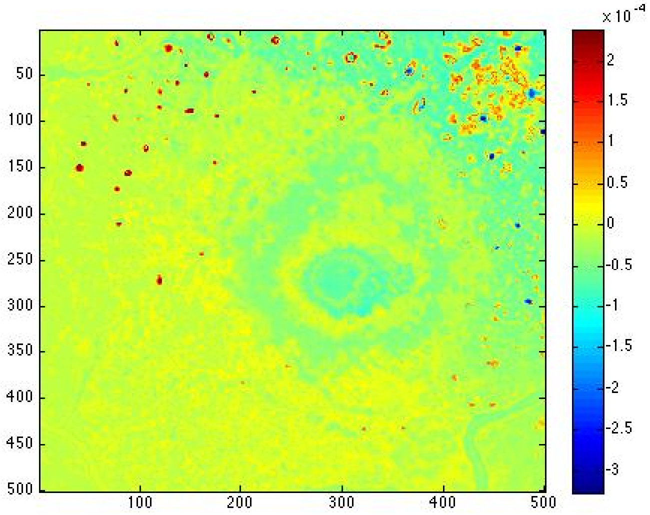


Pre-Drusen Images



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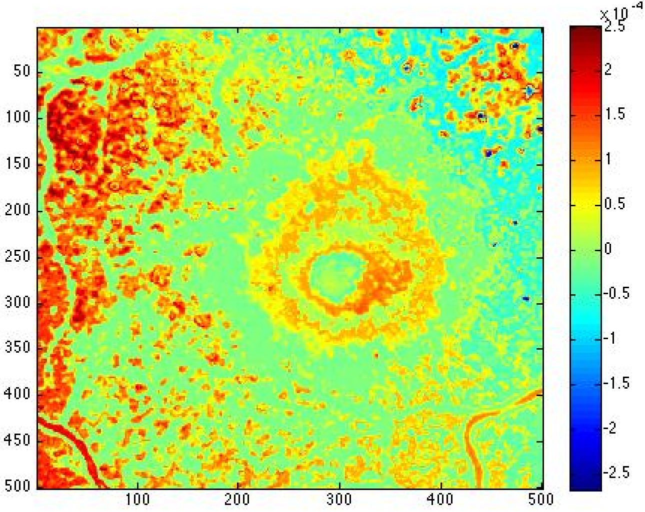
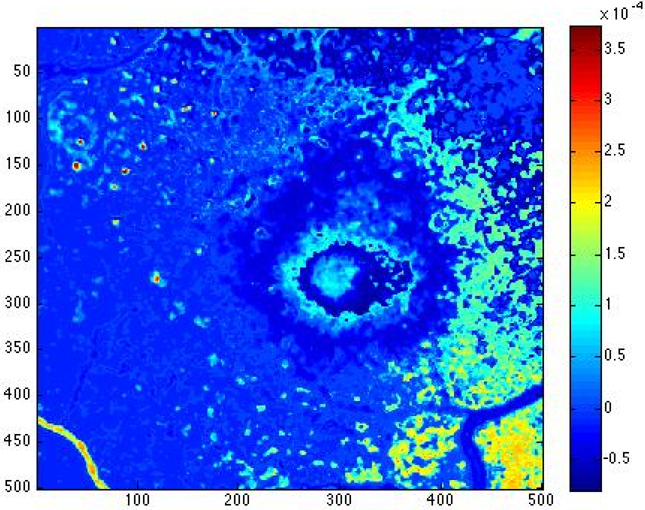
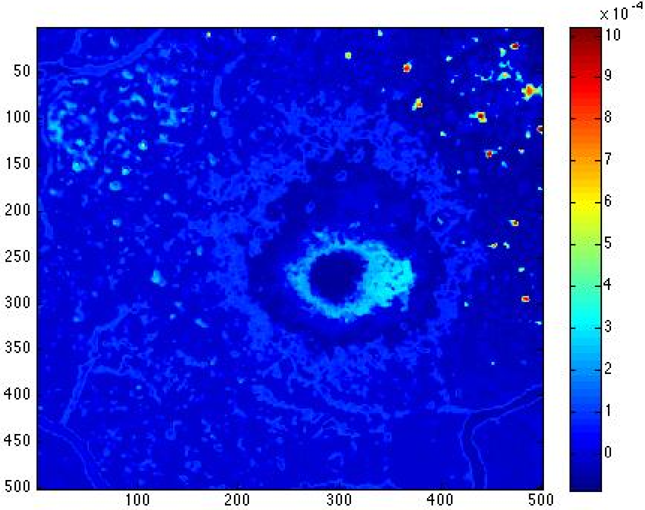


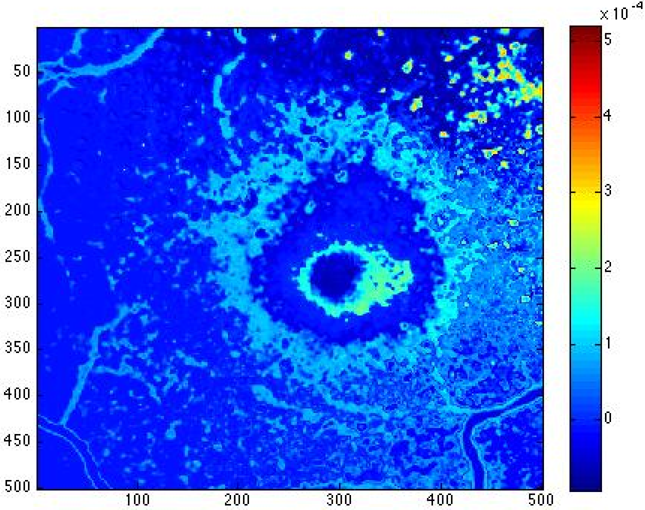
Pre-Drusen Images



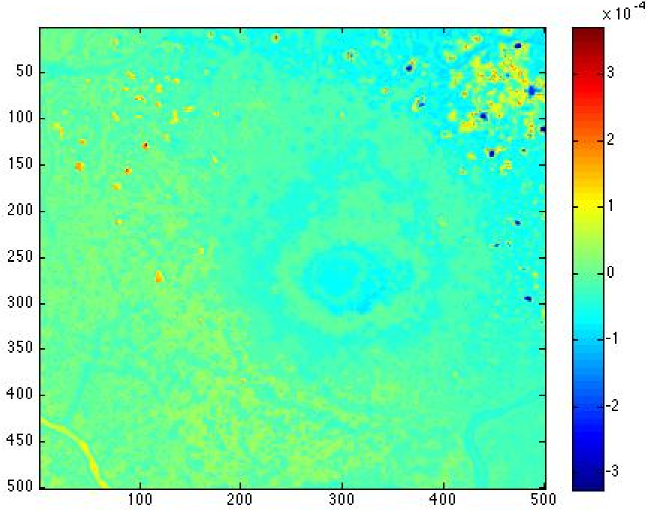
Pre-Drusen Images



Pre-Drusen Images



Pre-Drusen Images



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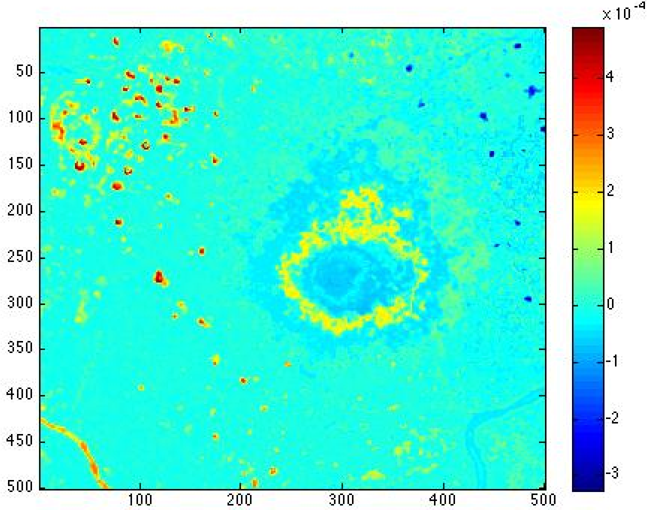

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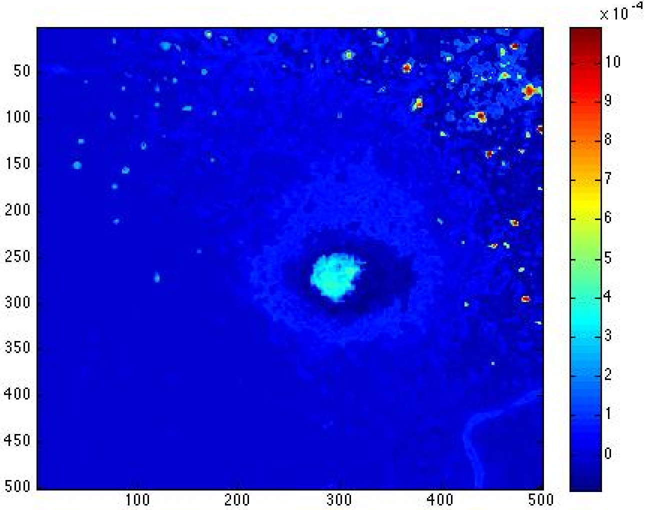

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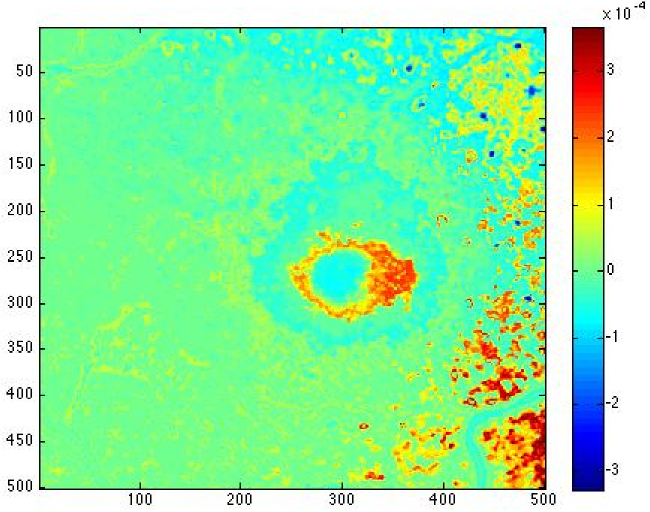

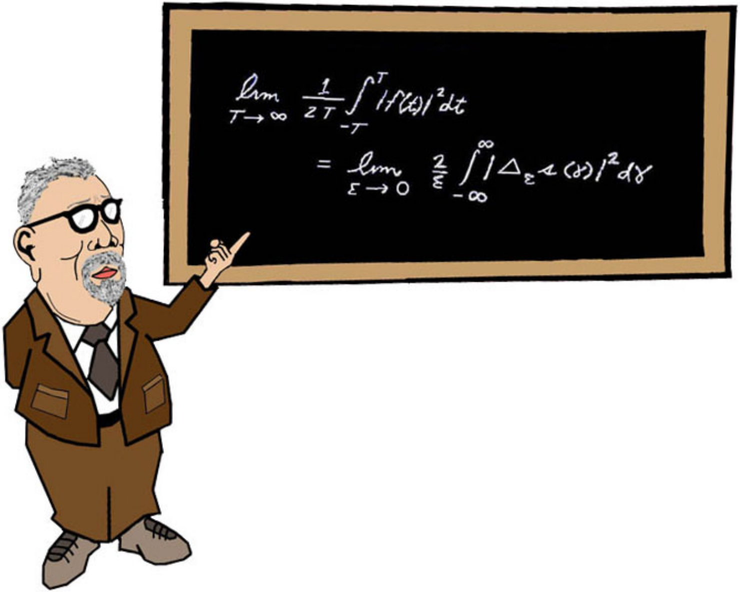
$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 dt$
$= \lim_{\varepsilon \to 0} \frac{2}{\varepsilon} \int_{-\infty}^{\infty} |\Delta_\varepsilon s(\gamma)|^2 d\gamma$

Abstract

A significant underlying component for the effective design of phase-coded waveforms is the construction of finite unimodular codes whose autocorrelations are zero everywhere except at the dc-component. We refer to such codes as CAZACs, Constant Amplitude Zero AutoCorrelation codes.

We begin by describing some known results in the long history of this subject. Then we construct new CAZACs and show that there is an infinitude of distinct CAZACs. This is important in the realm of waveform diversity, especially as regards a fine local analysis of the ambiguity function and the solutions of both the narrow band and wide band radar ambiguity problems.

We also present the vector-valued theory as well as constructions of infinite CAZAC codes.

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Accomplishments

- Developed libraries of CAZAC codes parameterized by design variables, proven mathematically and made available by user friendly software (CAZAC Playstation).
- Refined and formulated new, large classes of quadratic phase CAZAC codes and introduced Björck CAZAC codes to achieve diverse discrete periodic ambiguity function behavior.
- Enhanced sidelobe suppression by averaging and mixing techniques for CAZAC codes.
- Constructed vector-valued CAZAC codes with frame properties. This was motivated by the fact that frames lead to robust/stable signal decompositions. Vector-valued CAZAC codes are relevant in light of vector sensor and MIMO capabilities.
- Established the theory of waveforms coded by finite Gabor systems, and made a quantitative comparison with the non-Gabor case (A. Bourouihiya).
- Proved preliminary mathematical results to estimate the number of essentially different CAZAC codes of length N.

Transition and the future

- Our CAZAC software continues to be developed. This is ongoing work in order to develop a useful tool for the community. See

  www.math.umd.edu/~jjb/cazac/

- We shall analyze the wideband radar ambiguity function,

$$WA(u)(x,a) = \sqrt{a} \int u(a(t-x))\overline{u(t)}dt,$$

  in terms of wavelet frames, with the intent of solving the wideband radar ambiguity problem.

- We intend to complete our geometrical analysis of Shapiro-Rudin polynomials, and to extend the study to Golay pairs.

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Transition and the future

- We shall further develop and implement our theory of vector-valued ambiguity functions in terms of our notion of frame multiplication and the role of finite groups.
- Our previous MURI results on number theoretic CAZAC codes, such as Björck codes, serving as coefficients for phase-coded waveforms, will be analyzed in the vector-valued setting.
- We shall construct alternatives to the Golay waveform modality by means of our vector-valued theory.
- Gabor frames and pseudodifferential operators will be incorporated in our investigation of the narrow band radar ambiguity function
- We are using our frame potential characterization of FUNTFs in conjunction with L^1-sparse representation criteria in order to construction a new quantization scheme, called SRQP (Sparse Representation Quantization Procedure), which goes beyond $\Sigma - \Delta$.

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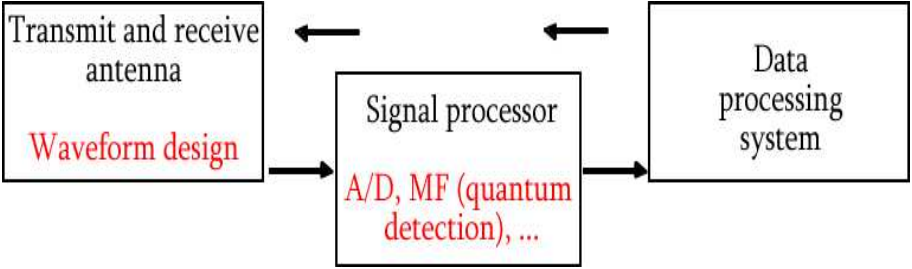
Immediate future

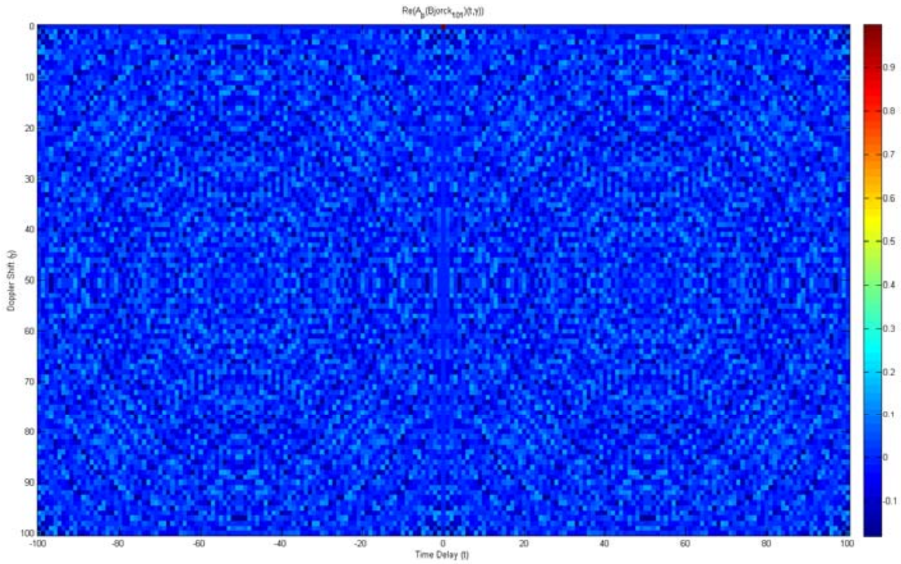
Integrate systematic dimension reduction in terms of non-linear kernel methods and Frame Potential Energy with the sparse coefficient set approach just developed. This kernel-frame approach was reported on at the Adelphi ARO-MURI 2010 meeting.

Processing

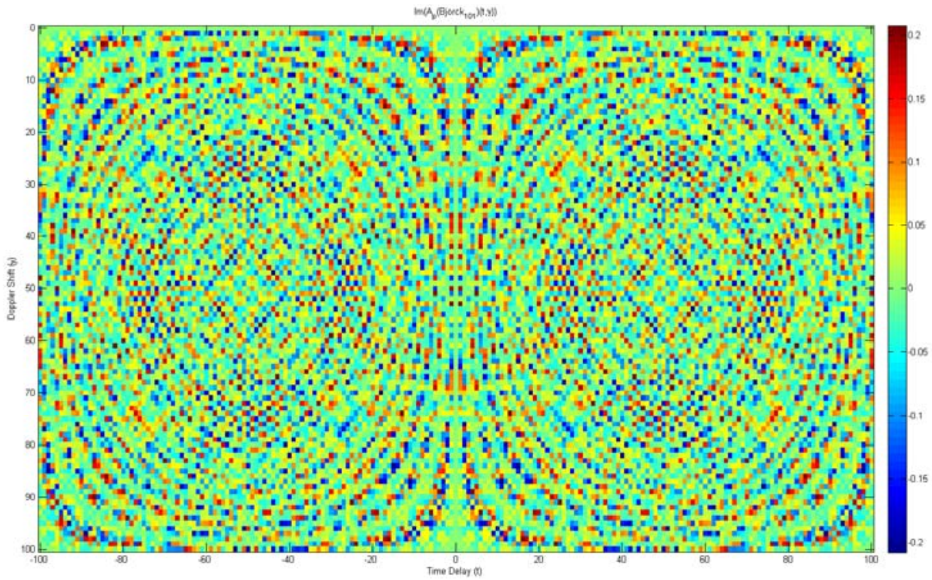


Transmit and receive antenna
Waveform design
Signal processor
A/D, MF (quantum detection), ...
Data processing system

Bjorck Code of Length 101



Real Part



Imaginary Part

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Waveform design and balayage

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Balayage, Fourier frames, and sampling theory

Balayage and the theory of generalized Fourier frames

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FRAMES

Definition

A collection $(e_n)_{n\in\Lambda}$ in a Hilbert space $\mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist $0 < A \leq B < \infty$ such that

$$\forall x \in \mathcal{H},\; A\|x\|^2 \leq \sum_{n\in\Lambda} |\langle x, e_n\rangle|^2 \leq B\|x\|^2.$$

The constants A and B are the *frame bounds*. If $A = B$, the frame is an *A-tight* frame.

Frames

Definition

- Bessel (analysis) operator L: $\mathcal{H} \to \ell^2(\Lambda)$

$$Lx = (\langle x, e_n \rangle)$$

- Synthesis operator L^*, the Hilbert space adjoint of L
- Frame operator $S = L^*L : \mathcal{H} \to \mathcal{H}$,

$$Sx = \sum \langle x, e_n \rangle e_n.$$

By the definition of frames, S satisfies $AI \leq S \leq BI$.

- Grammian operator $G = LL^* : \ell^2(\Lambda) \to \ell^2(\Lambda)$.

Frames

Definition

A collection $\{e_n\}_{n\in J}$ in a Hilbert space $\mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist $0 < A \leq B < \infty$ such that

$$\forall x \in \mathcal{H}, \quad A\|x\|^2 \leq \sum_{n\in J} |\langle x, e_n\rangle|^2 \leq B\|x\|^2.$$

- *Bessel (analysis) operator* $L: \ \mathcal{H} \to \ell^2(J)$

$$\forall x \in \mathcal{H}, \quad Lx = \{\langle x, e_n\rangle\}_{n\in J}$$

- *Synthesis operator* $L^*: \ \ell^2(J) \to \mathcal{H}$,

$$\forall c = (c_n)_{n\in J} \in \ell^2(J), \quad L^*c = \sum_{n\in J} c_n e_n.$$

- *Frame operator* $S = L^*L$, and *Grammian operator* $G = LL^*$

Frames

$AI \leq S \leq BI$ implies that S is invertible and that $B^{-1}I \leq S^{-1} \leq A^{-1}I$.

Definition

Let $F = \{e_n\}$ be a frame, and let $\widetilde{e}_n = S^{-1}e_n$. $\widetilde{F} = \{\widetilde{e}_n\}$ is the *dual frame* of F.

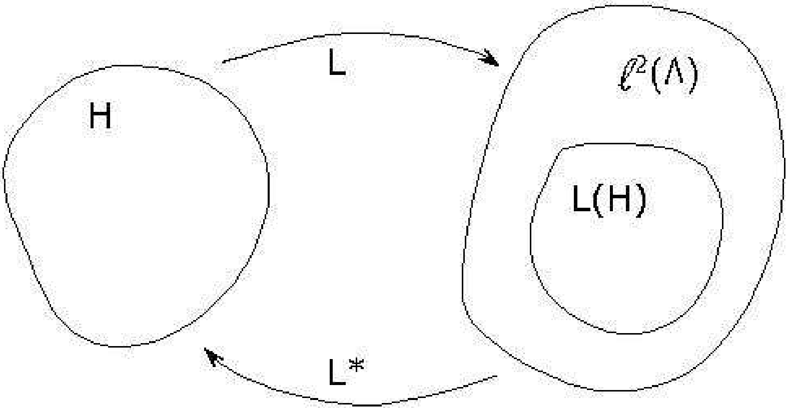
- $\sum \langle x, e_n\rangle \widetilde{e}_n = S^{-1}\sum \langle x, e_n\rangle e_n = S^{-1}Sx = x$.
- $\sum \langle x, \widetilde{e}_n\rangle e_n = \sum \langle S^{-1}x, e_n\rangle e_n = SS^{-1}x = x$.
- The frame operator of $\widetilde{F}$ is S^{-1} since

$$\sum \langle x, \widetilde{e}_n\rangle \widetilde{e}_n = S^{-1}\sum \langle S^{-1}x, e_n\rangle e_n = S^{-1}SS^{-1}x = S^{-1}x.$$

- $\sum |\langle x, \widetilde{e}_n\rangle|^2 = \langle S^{-1}x, x\rangle$. Then,

$$B^{-1}\|x\|^2 \leq \sum |\langle x, \widetilde{e}_n\rangle|^2 \leq A^{-1}\|x\|^2.$$

Frames



Frames

Theorem

Let H be a Hilbert space.

$$\{e_n\}_{n\in\Lambda} \subseteq H \text{ is } A\text{-tight} \Leftrightarrow S = AI,$$

where I is the identity operator.

Proof. $(\Rightarrow)$ If $S = L^*L = AI$, then $\forall x \in H$

$$\begin{aligned} A\|x\|^2 &= A\langle x, x\rangle = \langle Ax, x\rangle = \langle Sx, x\rangle \\ &= \langle L^*Lx, x\rangle = \langle Lx, Lx\rangle \\ &= \|Ly\|^2_{l^2(\Lambda)} \\ &= \sum_{i\in\Lambda} |\langle x, e_i\rangle|^2. \end{aligned}$$

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Frames

Proof. ($\Leftarrow$) If $\{e_i\}_{i\in\Lambda}$ is A-tight, then $\forall x \in H$, $A\langle x, x\rangle$ is

$$A\|x\|^2 = \sum_{i\in K} |\langle x, e_i\rangle|^2 = \sum_{i\in K} \langle x, e_i\rangle\langle e_i, x\rangle = \left\langle \sum_{i\in K} \langle x, e_i\rangle e_i, x \right\rangle = \langle Sx, x\rangle.$$

Therefore,

$$\forall x \in H, \quad \langle (S - AI)x, x\rangle = 0.$$

In particular, $S - AI$ is Hermitian and positive semi-definite, so

$$\forall x, y \in H, \quad |\langle (S - AI)x, y\rangle| \leq \sqrt{\langle (S - AI)x, x\rangle\langle (S - AI)y, y\rangle} = 0.$$

Thus, $(S - AI) = 0$, so, $S = AI$.

Frames

Theorem (Vitali, 1921)

Let H be a Hilbert space, $\{e_n\} \subseteq H$, $\|e_n\| = 1$.

$$\{e_n\} \text{ is 1-tight} \Leftrightarrow \{e_n\} \text{ is an ONB.}$$

Proof. If $\{e_n\}$ is 1-tight, then $\forall y \in H$

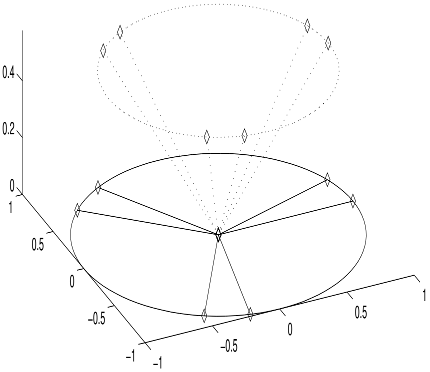
$$\|y\|^2 = \sum_n |\langle y, e_n\rangle|^2.$$

Since each $\|e_n\| = 1$, we have

$$1 = \|e_n\|^2 = \sum_k |\langle e_n, e_k\rangle|^2 = 1 + \sum_{k, k\neq n} |\langle e_n, e_k\rangle|^2$$

$$\Rightarrow \sum_{k\neq n} |\langle e_n, e_k\rangle|^2 = 0 \Rightarrow \forall n \neq k,\ \langle e_n, e_k\rangle = 0$$

Recent applications of FUNTFs



Naimark Theorem

Definition

Let H be a Hilbert space, $V \subseteq H$ a closed subspace, and

$$V^{\perp} = \{z \in H : \forall y \in V,\ \langle z, y \rangle = 0\}$$

be its orthogonal complement. Then, for every $x \in H$, there is a unique $y \in V$ satisfying

$$\|x - y\| = \min\{\|x - y'\| : y' \in V\},$$

and a unique $z \in V^{\perp}$ such that $x = y + z$.
The map $P_V : H \to V$, $P_V x = y$ is the *orthogonal projection* on V.

If $\{v_n\}$ is an orthonormal basis for V, then P_V can be expressed as

$$\forall x \in H, \quad P_V x = \sum_n \langle x, v_n \rangle v_n.$$

Naimark Theorem

Can we make tight frames for $H = \mathbb{F}^d$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) with prescribed redundancy?

Yes. Take an $N \times N$ unitary matrix U, and choose any d columns of it to form an $N \times d$ matrix L. Then, $L^* L = I$, which means, the rows of L form a 1-tight frame for $\mathbb{F}^d$.

How about FUNTFs?

Yes, we shall explain how to generate FUNTFs by using the frame potential.

Naimark Theorem

If $\{e_n\}_{n=1}^N$ is an A-tight frame for $\mathbb{F}^d$, and L is its Bessel map, then $L^*L = AI$, i.e., the set of the columns of L, $\{c_1, \dots, c_d\}$ is a $\sqrt{A}$-normed orthogonal set in $\mathbb{F}^N$. Let $V = span\{c_1, \dots, c_d\}$, and let $\{c_{d+1}, \dots, c_N\}$ be a $\sqrt{A}$-normed orthogonal basis for $V^\perp$. Then, the matrix

$$U = A^{-1/2}[c_1 \dots c_N]$$

is a unitary matrix, since its columns give an ONB for $\mathbb{F}^d$. Then, the rows of U also give an ONB for $\mathbb{F}^d$. Let $\widetilde{e}_k$ be the kth row of $A^{1/2}U$. Then,

1. $\{\widetilde{e}_k\}$ is a $\sqrt{A}$-normed orthogonal basis for $\mathbb{F}^N$,
2. $e_k = P\widetilde{e}_k$, where $P : \mathbb{F}^N \to \mathbb{F}^d$,

$$P(x[1], \dots x[N]) = (x[1], \dots, x[d]).$$

Naimark Theorem

Theorem (Naimark)

Let H be a d-dimensional Hilbert space, $\{e_n\}_{n=1}^N$ be an A-tight frame for H. Then there exists an N-dimensional Hilbert space $\widetilde{H}$, and orthogonal A-normed set $\{\widetilde{e}_n\}_{n=1}^N \subseteq \widetilde{H}$ such that

$$P_H \widetilde{e}_n = e_n$$

where P_H is the orthogonal projection onto H.

Hadamard matrices and infinite CAZAC codes

Rationale and theorem

Example

a. Let N be odd and let $\omega = e^{2\pi i/N}$. Then, $u(k) = \omega^{k^2}$, $0 \leq k \leq N-1$, is a CAZAC waveform. By the Corollary, $|A_u(m,n)| = |\omega^{m^2}| = 1$ if $2m+n = l_{m,n}N$ for some $l_{m,n} \in \mathbb{Z}$ and $|A_u(m,n)| = 0$ otherwise, i.e., $A_u(m,n) = 0$ on $\mathbb{Z}_N \times \mathbb{Z}_N$ unless $2m+n \equiv 0 \bmod N$. In the case $2m+n = l_{m,n}N$ for some $l_{m,n} \in \mathbb{Z}$, we have the following phenomenon.

Rationale and theorem

Example (Continued)

If $0 \leq m \leq \frac{N-1}{2}$ and $2m + n = l_{m,n}N$ for some $l_{m,n} \in \mathbb{Z}$, then n is odd; and if $\frac{N+1}{2} \leq m \leq N-1$ and $2m + n = l_{m,n}N$ for some $l_{m,n} \in \mathbb{Z}$, then n is even. Thus, the values (m, n) in the domain of the discrete periodic ambiguity function A_u, for which $A_u(m, n) = 0$, appear as two parallel discrete lines. The line whose domain is $0 \leq m \leq \frac{N-1}{2}$ has odd function values n; and the line whose domain is $\frac{N+1}{2} \leq m \leq N-1$ has even function values n.

Rationale and theorem

Example

b. The behavior observed in (a) has extensions for primitive and non-primitive roots of unity.

Let $u : \mathbb{Z}_N \to \mathbb{C}$ be a Wiener waveform. Thus, $u(k) = \omega^{k^2}$, $0 \le k \le N-1$, and $\omega = e^{2\pi i j/M}$, $(j, M) = 1$, where M is defined in terms of N in Theorem 1. By the Corollary, for each fixed $n \in \mathbb{Z}_N$, the function $A_u(\bullet, n)$ of m vanishes everywhere except for a *unique* value $m_n \in \mathbb{Z}_N$ for which $|A_u(m_n, n)| = 1$.

Rationale and theorem

Example (Continued)

The hypotheses of Theorem 2 do not assume that $e^{2\pi ij/M}$ is a primitive Mth root of unity. In fact, in the case that $e^{2\pi ij/M}$ is *not* primitive, then, for certain values of n, $A_u(\bullet, n)$ will be identically 0 and, for certain values of n, $|A_u(\bullet, n)| = 1$ will have several solutions. For example, if $N = 100$ and $j = 2$, then, for each odd n, $A_u(\bullet, n) = 0$ as a function of m. If $N = 100$ and $j = 3$, then $(100, 3) = 1$ so that $e^{2\pi i3/100}$ is a primitive 100th root of unity; and, in this case, for each $n \in \mathbb{Z}_N$ there is a *unique* $m_n \in \mathbb{Z}_N$ such that $|A_u(m_n, n)| = 1$ and $A_u(m, n) = 0$ for each $m \neq m_n$.

Rationale and theorem

Proof.(Continued) Let N be odd, and set $u(k) = e^{2\pi i k^2/N}$. We calculate

$$\begin{aligned} A_{u_j}(m,n) &= \frac{1}{N}\sum_{k=0}^{N-1} u_j(m+k)\overline{u_j(k)}e^{2\pi i kn/N} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} e^{(2\pi i/N)(jm^2+2jkm+kn)} = e^{2\pi i jm^2/N}\frac{1}{N}\sum_{k=0}^{N-1} e^{2\pi i k(2jm+n)/N}. \end{aligned}$$

If $2jm + n \equiv 0 \bmod N$, then

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{2\pi i k(2jm+n)/N} = 1.$$

Otherwise, we have

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{2\pi i k(2jm+n)/N} = \frac{e^{2\pi i(2m+n)/N)N} - 1}{e^{(2\pi i(2m+n)/N} - 1} = 0.$$

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Rationale and theorem

Proof. Let N be even, and set $u_j(k) = e^{\pi ijk^2/N}$. We calculate

$$\begin{aligned} A_{u_j}(m,n) &= \frac{1}{N}\sum_{k=0}^{N-1} u_j(m+k)\overline{u_j(k)}e^{2\pi ikn/N} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} e^{(\pi i/N)(jm^2+2jkm+2kn)} = e^{\pi ijm^2/N}\frac{1}{N}\sum_{k=0}^{N-1} e^{2\pi ik(jm+n)/N}. \end{aligned}$$

If $jm+n \equiv 0 \bmod N$, then

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{2\pi ik(jm+n)/N} = 1.$$

Otherwise, we have

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{2\pi ik(jm+n)/N} = \frac{e^{(2\pi i(jm+n)/N)N} - 1}{e^{2\pi i(jm+n)/N} - 1} = 0.$$

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