Sampling in image representation and compression

Alfredo Nava-Tudela

Institute for Physical Science and Technology and Norbert Wiener Center, University of Maryland, College Park

Joint work with John J. Benedetto

Department of Mathematics and Norbert Wiener Center, University of Maryland, College Park

Overview

- Problem statement
- Image representation concepts
- Image compression basics
- Sparsity is the key, l_0-minimization, OMP
- Image compression revisited
- Imagery metrics
- Solving our problem: compressed sensing and *deterministic sampling masks*
- Solving our problem: results

Problem statement

Problem statement

Problem statement

JPEG, JPEG 2000

Sampling ⟷ Compression

Representation

Image representation concepts

Image representation concepts

$$I[n_1,n_2]$$

$$0 \leq n_1 < N_{1,}\ 0 \leq n_2 < N_2$$

Image representation concepts

$$I[n_1,n_2] = \text{pixel}$$

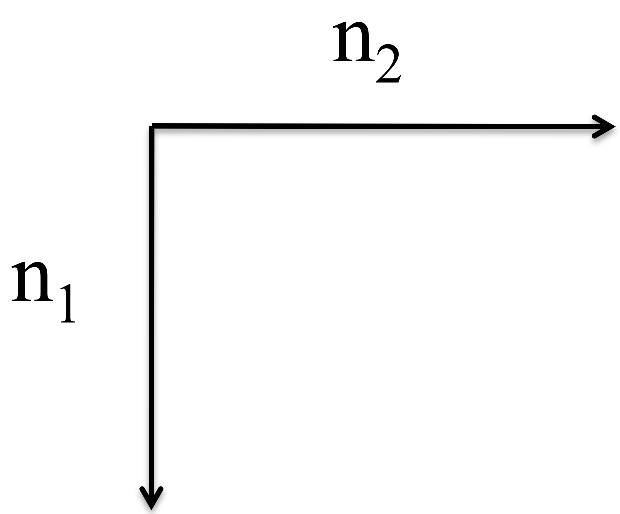


Image representation concepts

$I[n_1,n_2] \sim$ intensity, brightness at $[n_1,n_2]$

$I[n_1,n_2] \in \{0, \ldots , 2^B-1\}$, or

$I[n_1,n_2] \in \{-2^{B-1}, \ldots , 2^{B-1}-1\}$, where

$I[n_1,n_2] = \text{round}(2^B\, I'[n_1,n_2])$ and

$I'[n_1,n_2] \in [0,1)$ or $[-½, ½)$

Image representation concepts

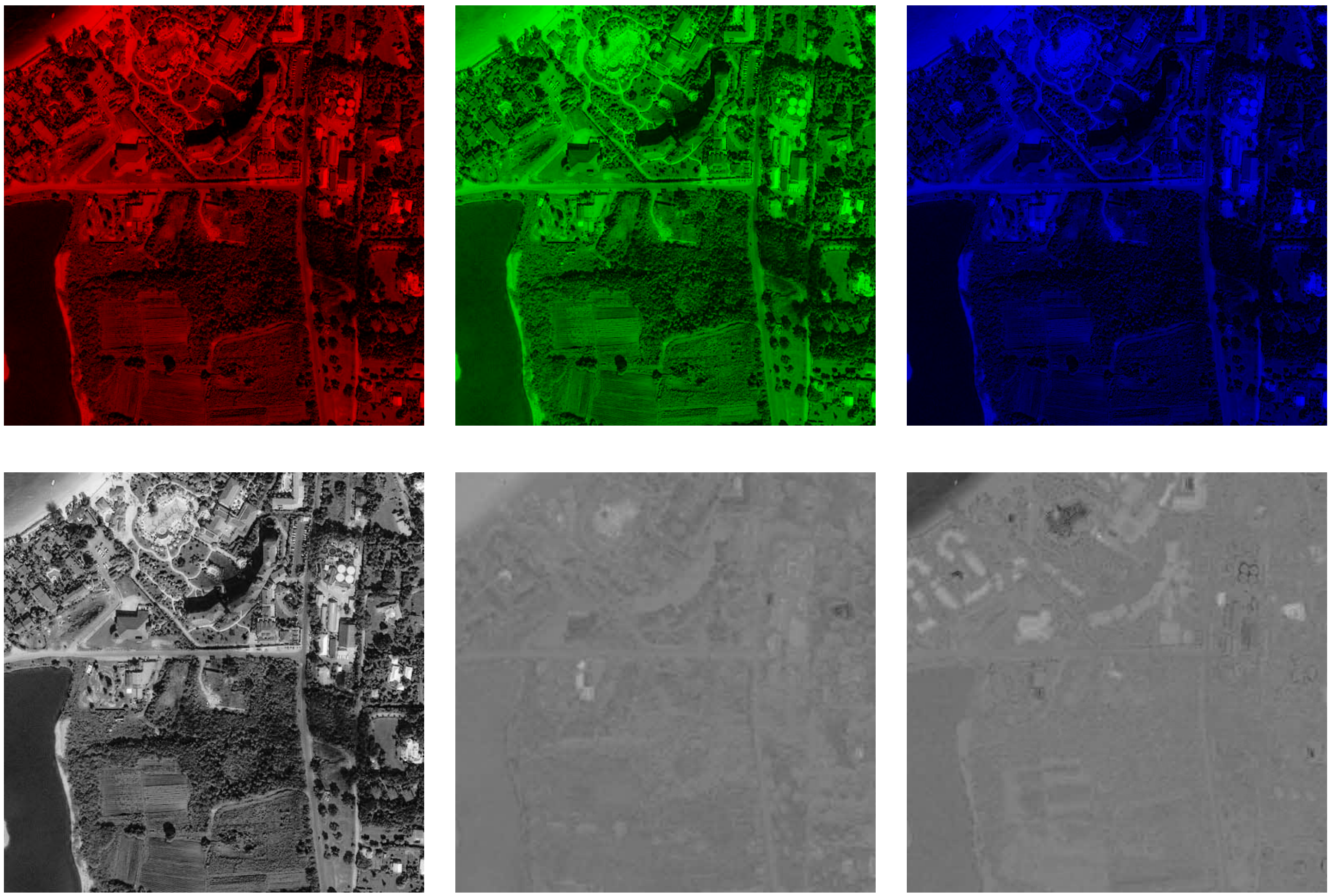


Image compression

$512 \times 512 \times 8 \times 3 = 6{,}291{,}456$ bits

Image compression

JPEG, JPEG 2000

Image compression

1) Partitioning of the image I in sub-images

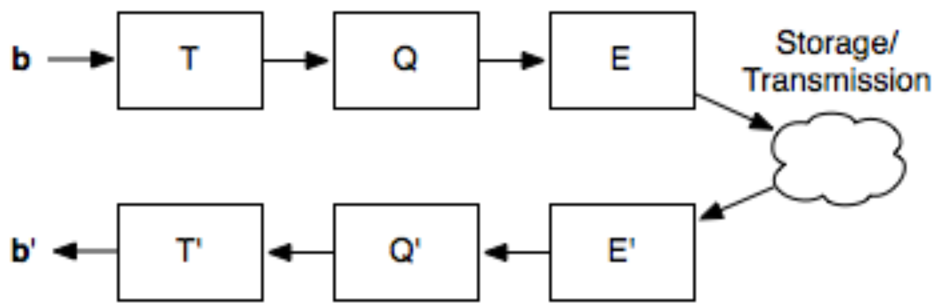
Image compression

1) Partitioning of the image I in sub-images
2) Transform sub-images to exploit correlations within them

Image compression

1) Partitioning of the image I in sub-images
2) Transform sub-images to exploit correlations within them
3) Quantize and encode

Image compression



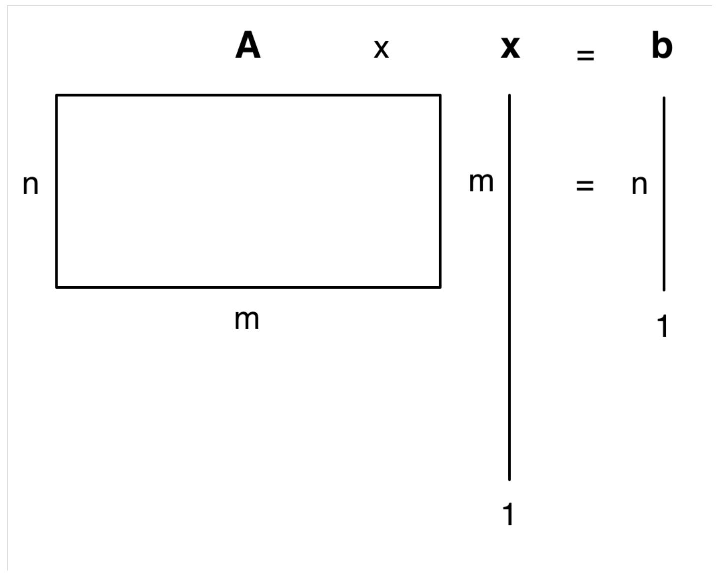
Sparsity is the key

Cn u rd ths?

vs

Can you read this?

Sparsity is the key



Sparsity

The l_0 "norm":

$$||\mathbf{x}||_0 = \# \{k : x_k \neq 0\}$$

l_0-minimization ~ sparse solution

$$(P_0): \quad \min_{\mathbf{x}} ||\mathbf{x}||_0 \text{ subject to } ||\mathbf{A}\mathbf{x} - \mathbf{b}||_2 = 0$$

l_0-minimization ~ sparse solution

$$(P_0^{\varepsilon}): \quad \min_{\mathbf{x}} \|\mathbf{x}\|_0 \text{ subject to } \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 < \varepsilon$$

l_0-minimization ~ sparse solution

(P_0^{ε}): $\min_{\mathbf{x}} ||\mathbf{x}||_0$ subject to $||\mathbf{A}\mathbf{x} - \mathbf{b}||_2 < \varepsilon$

Solving (P_0^{ε}) is NP-hard!
Is there any hope?

Finding sparse solutions:OMP

Orthogonal Matching Pursuit algorithm:

Task: Approximate the solution of (P_0) : $\min_{\mathbf{x}} \|\mathbf{x}\|_0$ subject to $\mathbf{Ax} = \mathbf{b}$.

Parameters: We are given the matrix $\mathbf{A}$, the vector $\mathbf{b}$, and the threshold ϵ_0.

Initialization: Initialize $k = 0$, and set

- The initial solution $\mathbf{x}^0 = \mathbf{0}$.
- The initial residual $\mathbf{r}^0 = \mathbf{b} - \mathbf{Ax}^0 = \mathbf{b}$.
- The initial solution support $\mathcal{S}^0 = Support\{\mathbf{x}^0\} = \emptyset$.

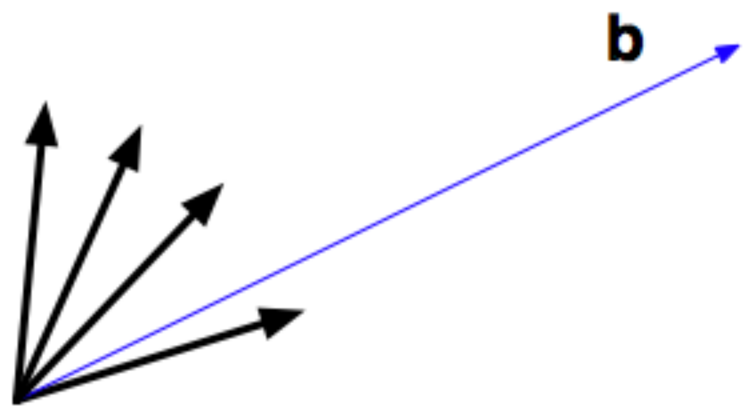
Main Iteration: Increment k by 1 and perform the following steps:

- **Sweep:** Compute the errors $\epsilon(j) = \min_{z_j} \|z_j \mathbf{a}_j - \mathbf{r}^{k-1}\|_2^2$ for all j using the optimal choice $z_j^* = \mathbf{a}_j^T \mathbf{r}^{k-1} / \|\mathbf{a}_j\|_2^2$.
- **Update Support:** Find a minimizer j_0 of $\epsilon(j)$: $\forall j \notin \mathcal{S}^{k-1}$, $\epsilon(j_0) \le \epsilon(j)$, and update $\mathcal{S}^k = \mathcal{S}^{k-1} \cup \{j_0\}$.
- **Update Provisional Solution:** Compute $\mathbf{x}^k$, the minimizer of $\|\mathbf{Ax} - \mathbf{b}\|_2^2$ subject to $Support\{\mathbf{x}\} = \mathcal{S}^k$.
- **Update Residual:** Compute $\mathbf{r}^k = \mathbf{b} - \mathbf{Ax}^k$.
- **Stopping Rule:** If $\|\mathbf{r}^k\|_2 < \epsilon_0$, stop. Otherwise, apply another iteration.

Output: The proposed solution is $\mathbf{x}^k$ obtained after k iterations.

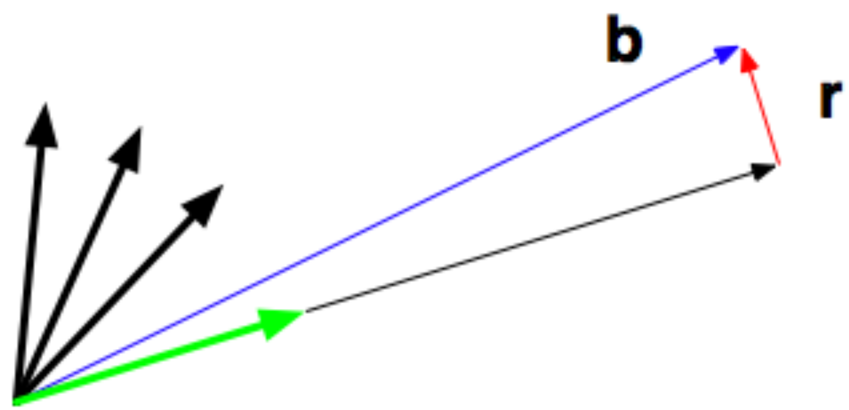
Finding sparse solutions:OMP

Orthogonal Matching Pursuit algorithm:



Finding sparse solutions:OMP

Orthogonal Matching Pursuit algorithm:



Finding sparse solutions:OMP

Orthogonal Matching Pursuit algorithm:

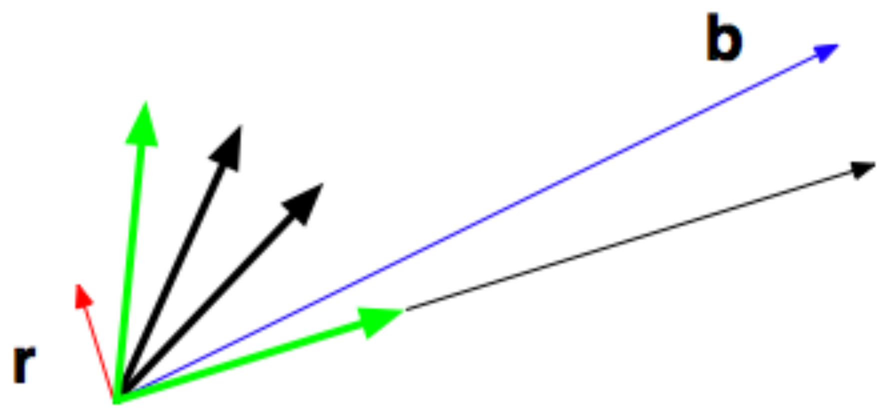
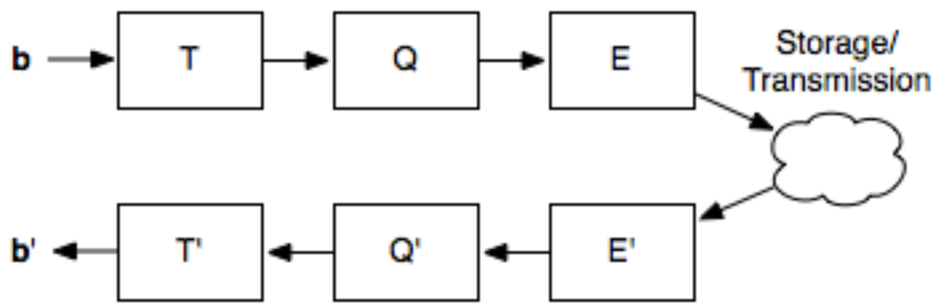
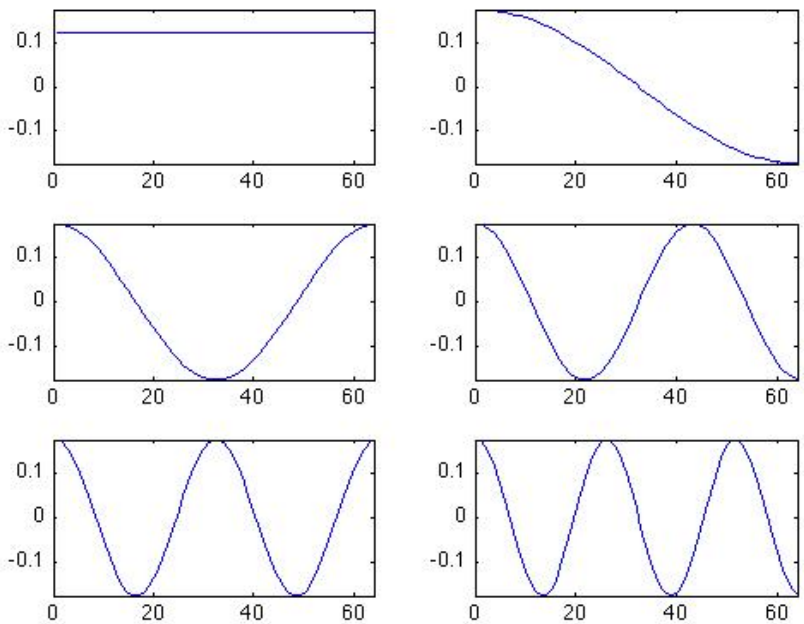


Image compression

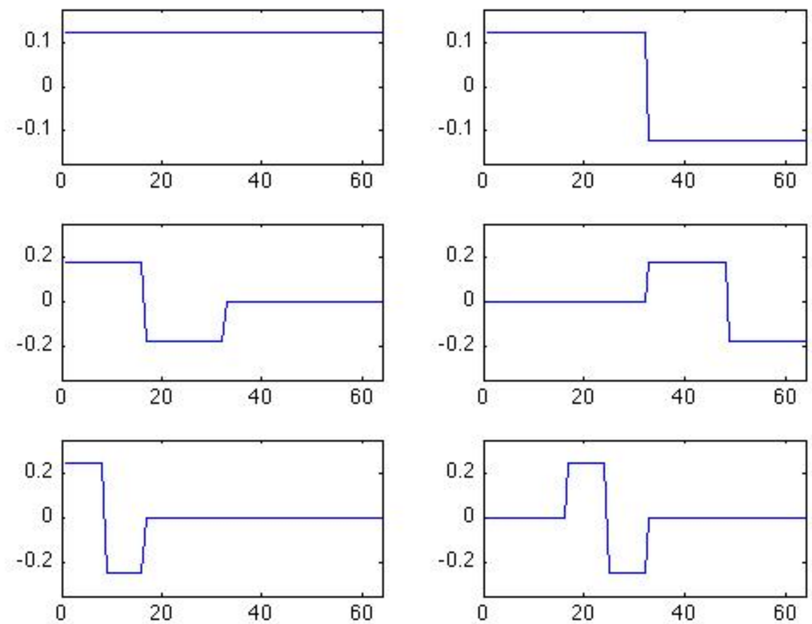


$\mathbf{T} = \mathbf{T}_{\varepsilon} = \mathrm{OMP}(\mathbf{A}, - ,\varepsilon)$, $\mathbf{T}' = \mathbf{A}$

We need a matrix **A**

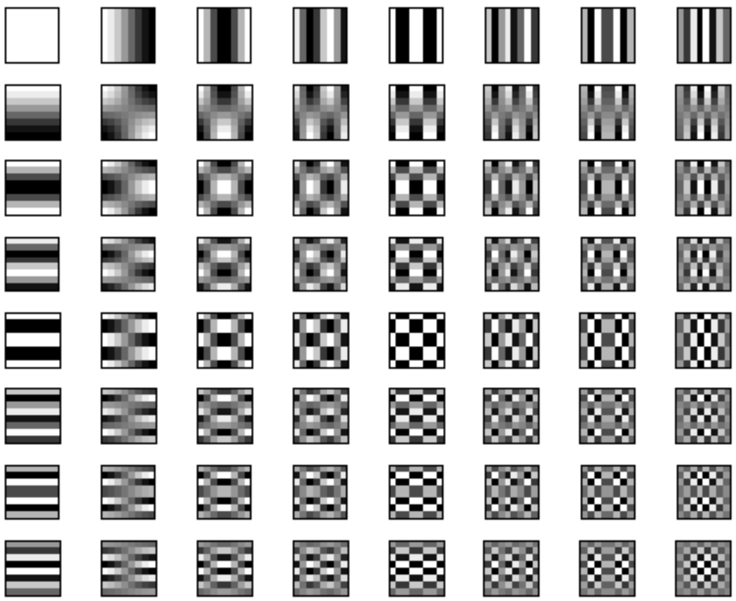


DCT

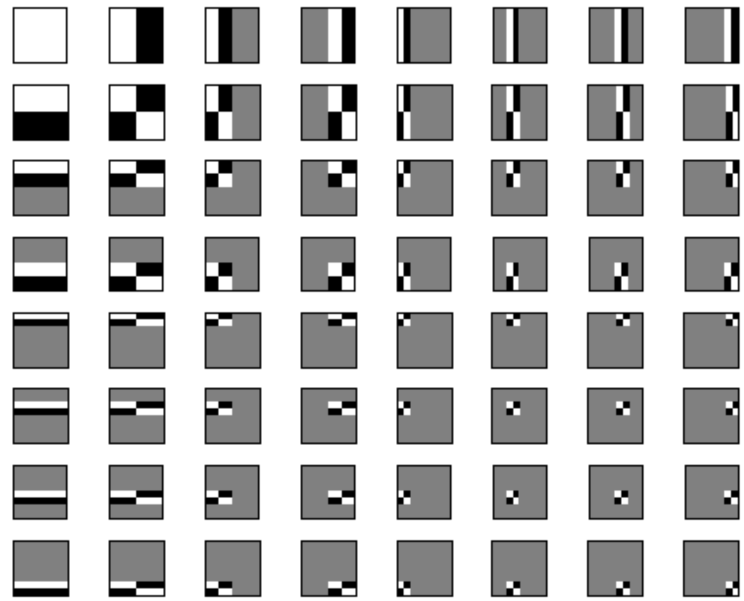


Haar

We need a matrix **A**

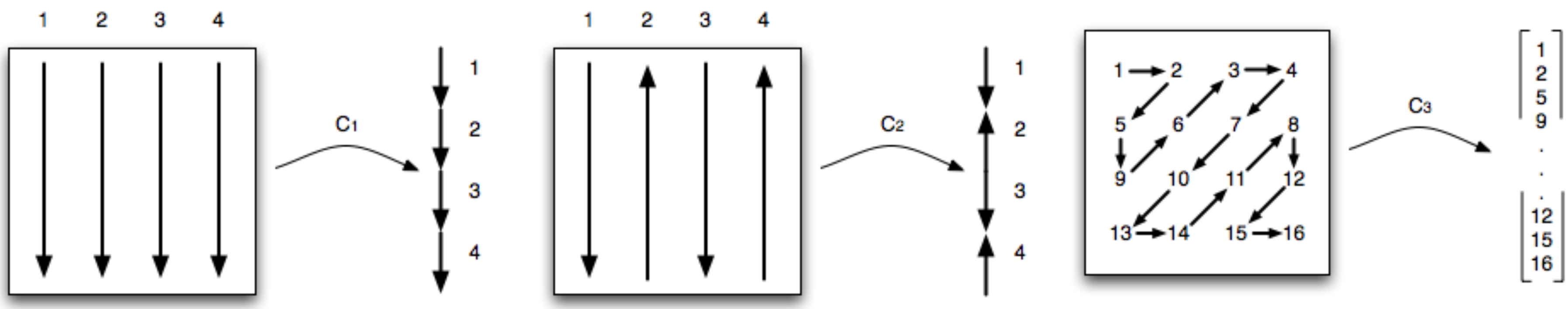


2D - DCT

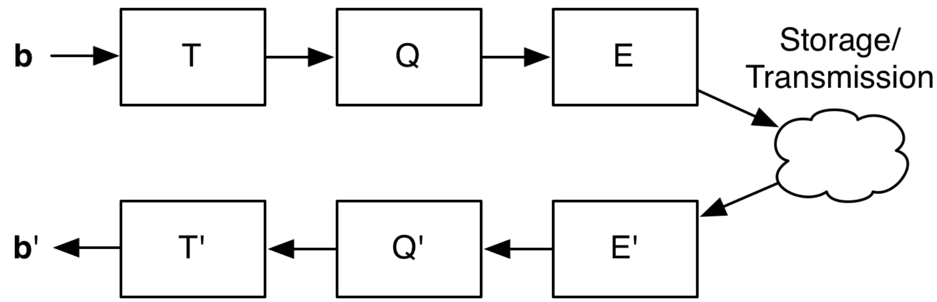


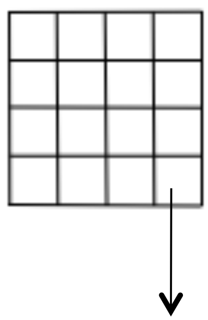
2D - Haar

We need a matrix **A**



Compressing a test image





$c_3(\square) = \mathbf{b}$

$\mathbf{x}_0 = \mathbf{T}_\varepsilon\, \mathbf{b} = \mathrm{OMP}(\mathbf{A}, \mathbf{b}, \varepsilon)$

$\boxtimes = c_3^{-1}(\mathbf{b}')$

$\mathbf{b}' = \mathbf{T}'\, \mathbf{x}_0 = \mathbf{A}\, \mathbf{x}_0$

Compressing a test image

□ ~ ☒ ? $\| \mathbf{b} - \mathbf{b}' \|_2 < \varepsilon$

But what does that mean visually?
How many bits were used?

Imagery metrics

Peak Signal-to-Noise Ratio (PSNR), measured in dB:

$$\mathrm{PSNR}(\mathbf{X},\mathbf{Y}) = 20 \log_{10}(\mathrm{MAX}_B / \sqrt{\mathrm{MSE}}),$$

with $\mathrm{MAX}_B = 2^B-1$, and $\mathrm{MSE} = \sum_{i,j} [\mathbf{X}(i,j) - \mathbf{Y}(i,j)]^2 / nm.$
In our case, $n = m = 512$, and B = 8, i.e. $\mathrm{MAX}_B = 255$.

Imagery metrics

Structural Similarity (SSIM), and Mean Structural Similarity(MSSIM) indices:

$$\mathrm{SSIM}(\mathbf{x},\mathbf{y}) = \frac{(2\,\mu_x\,\mu_y + C_1)\,(2\,\sigma_{xy} + C_2)}{(\mu_x^2 + \mu_y^2 + C_1)\,(\sigma_x^2 + \sigma_y^2 + C_2)}$$

$$\mathrm{MSSIM}(\mathbf{X},\mathbf{Y}) = \frac{1}{M}\sum_{j=1}^{M}\mathrm{SSIM}(\mathbf{x}_j,\mathbf{y}_j)$$

Imagery metrics

The normalized sparse bit-rate is

$$\mathrm{nsbr}(I,\mathbf{A},\varepsilon) = \sum ||\mathbf{x}_j||_0/N_1N_2,$$

where image I is of size N_1 by N_2.

Imagery metrics: test images

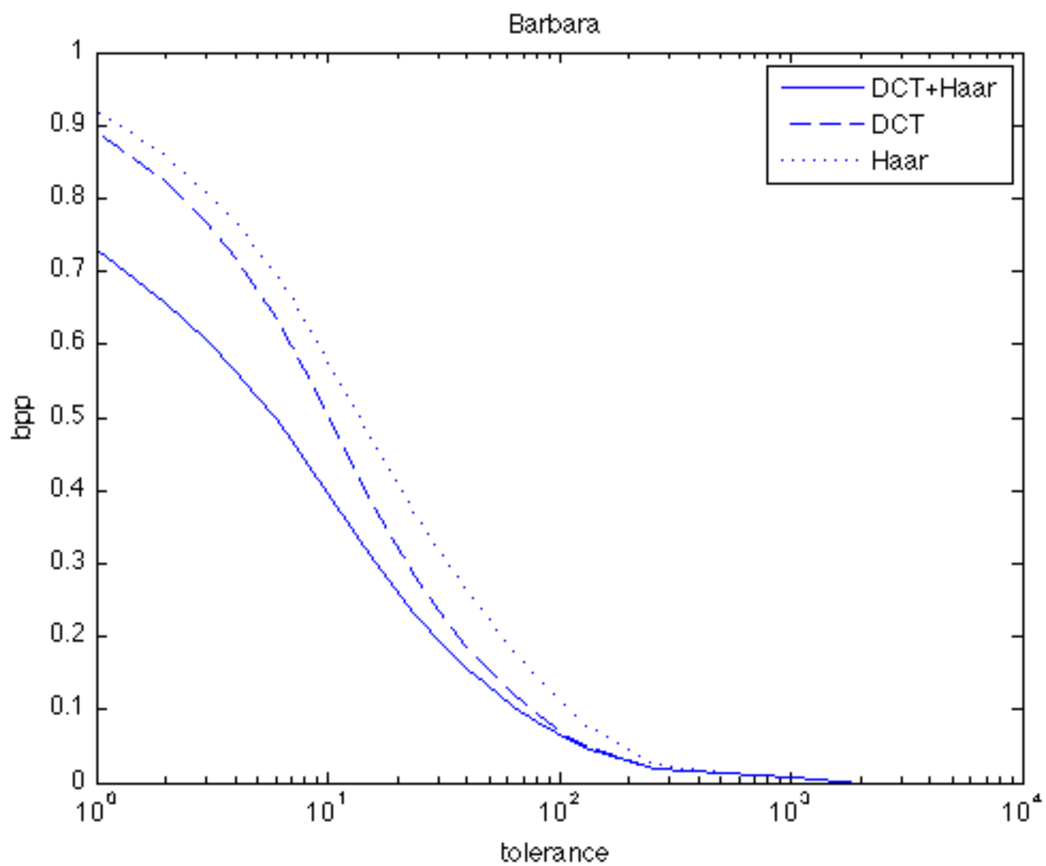
Barbara

Boat

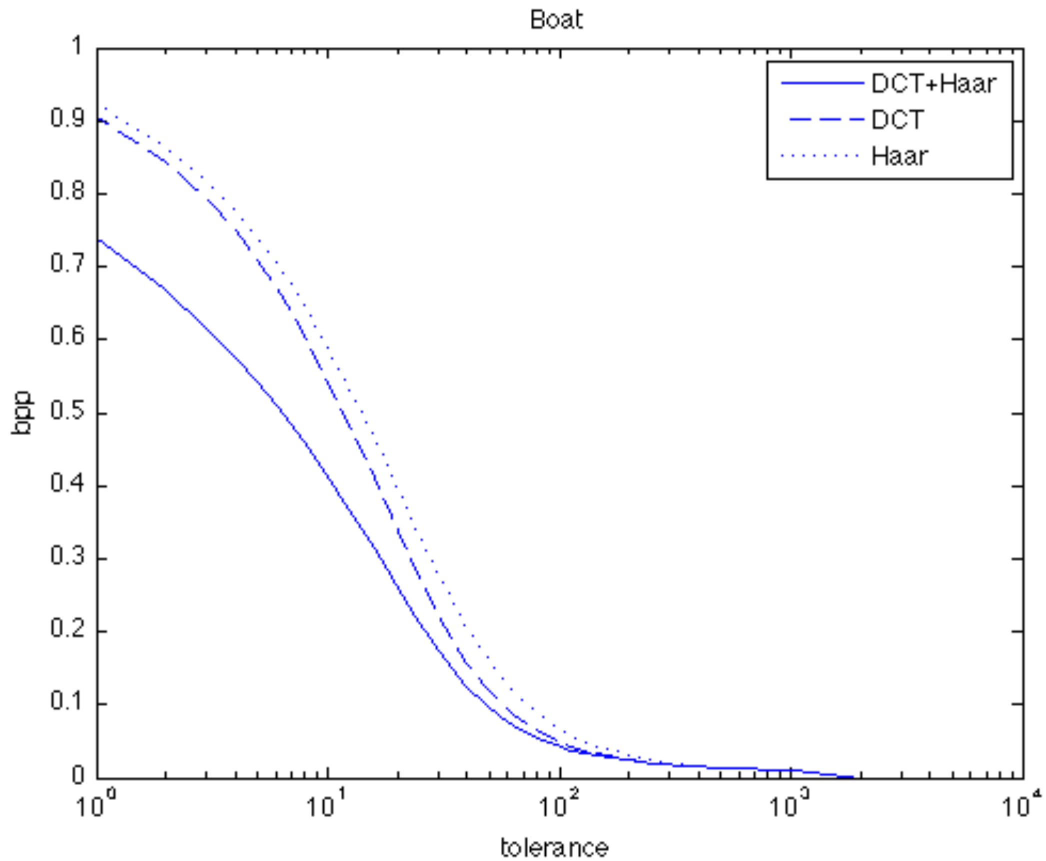
Elaine

Stream

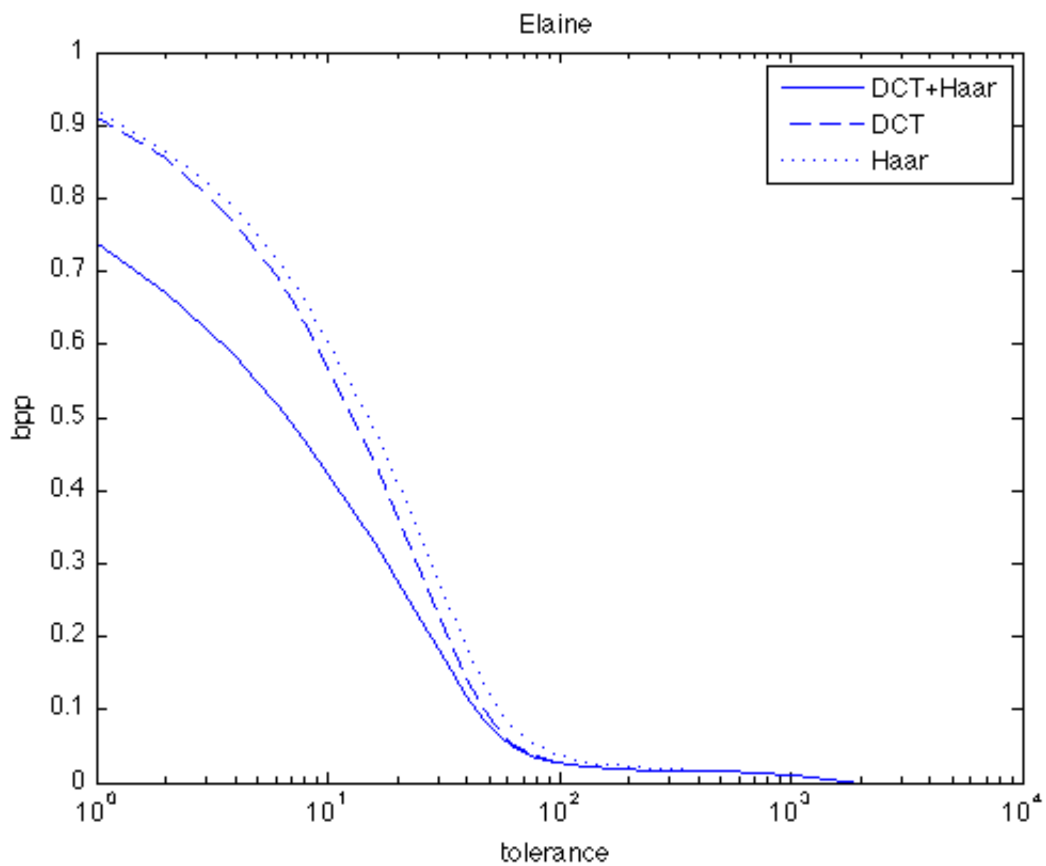
Imagery metrics: bpp vs ε



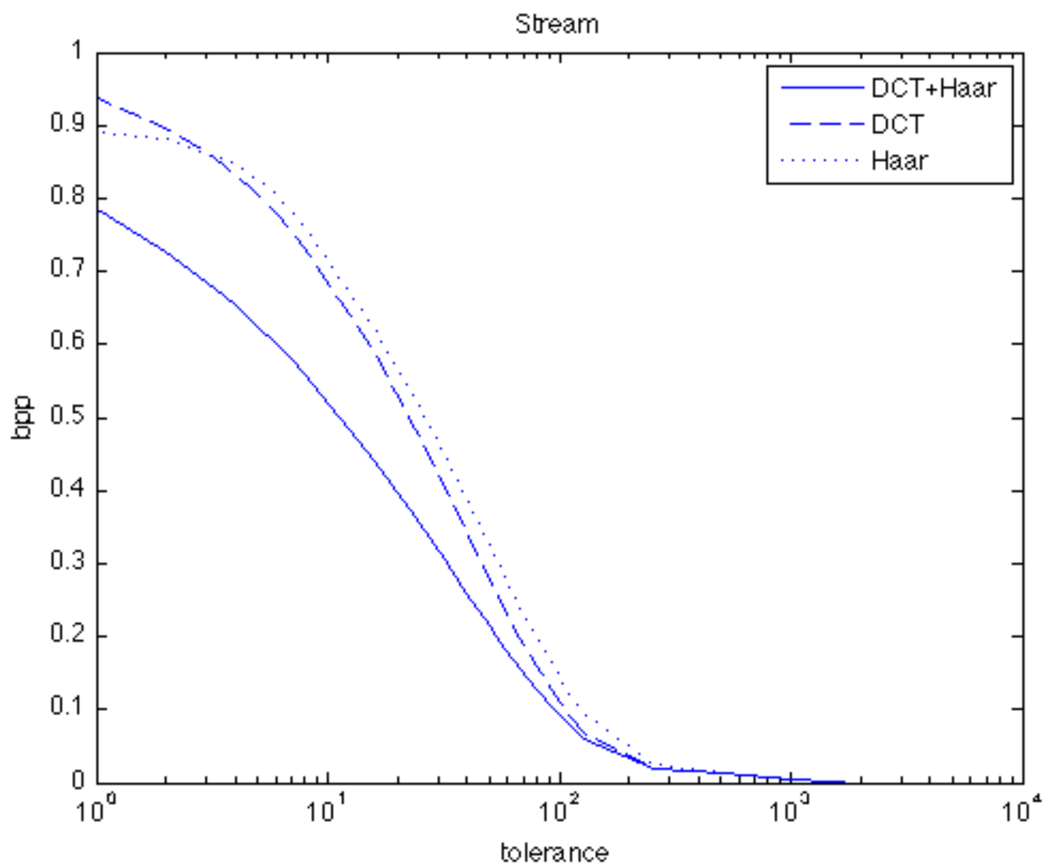
Imagery metrics: bpp vs ε



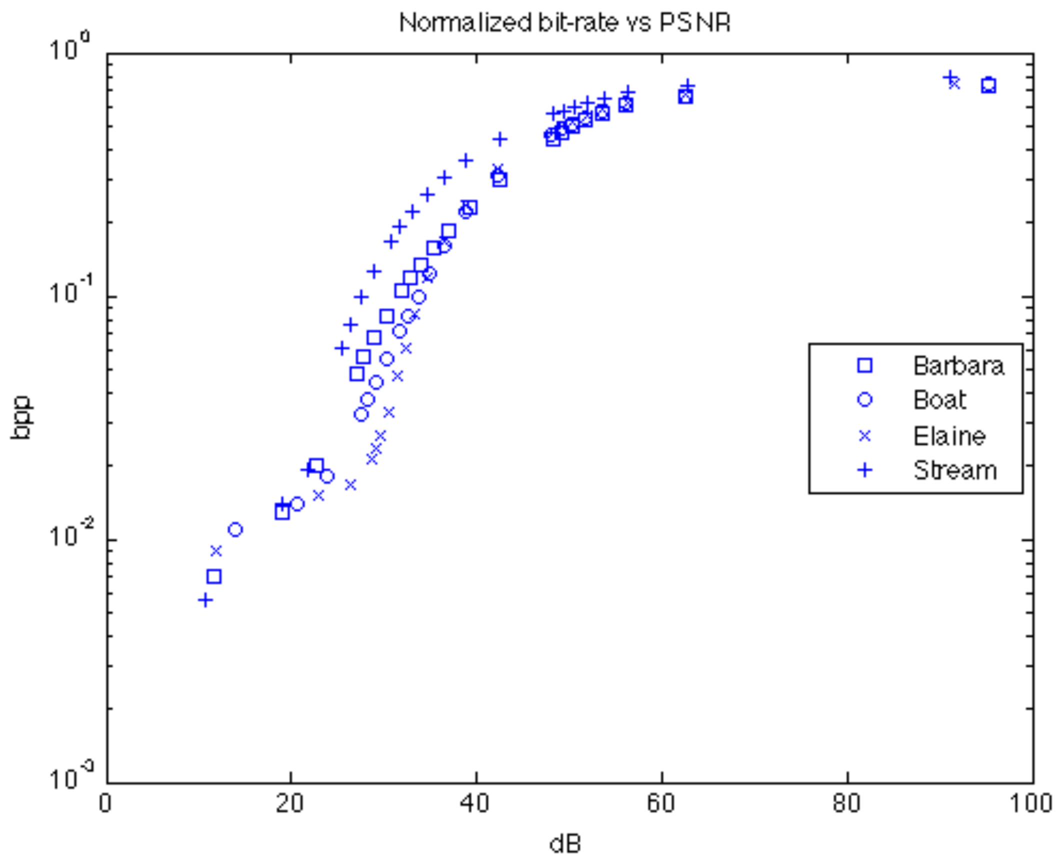
Imagery metrics: bpp vs ε



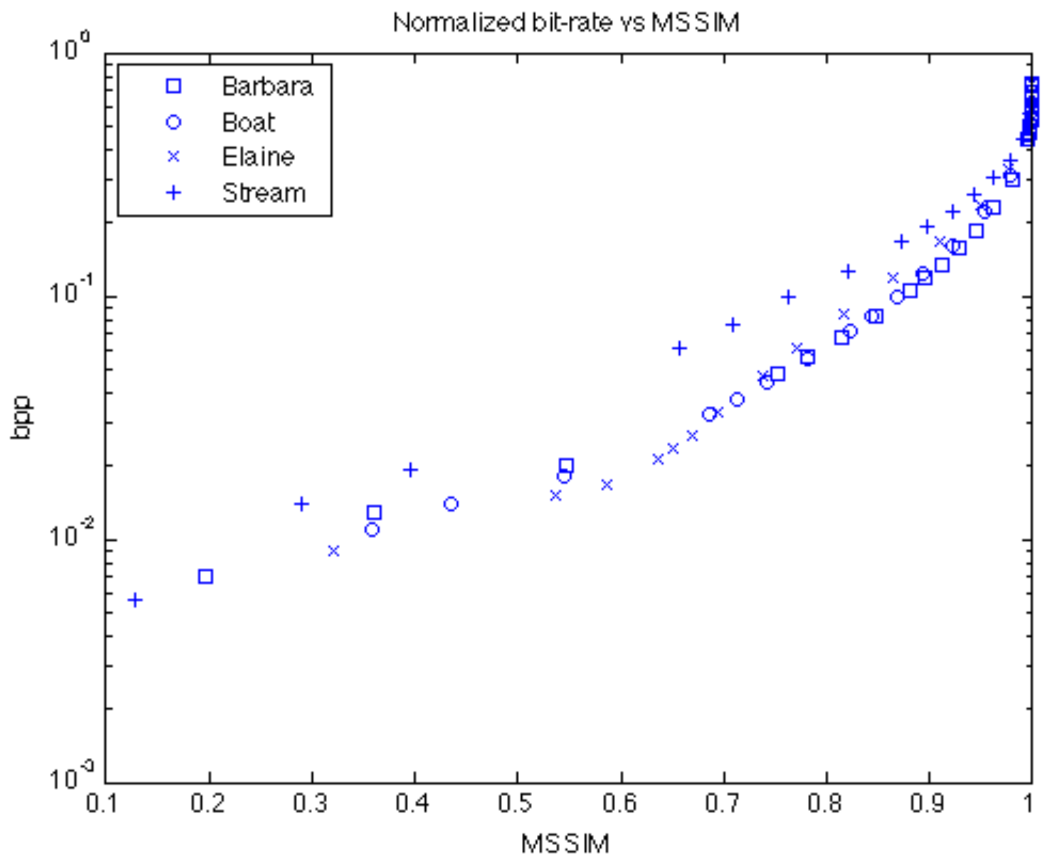
Imagery metrics: bpp vs ε



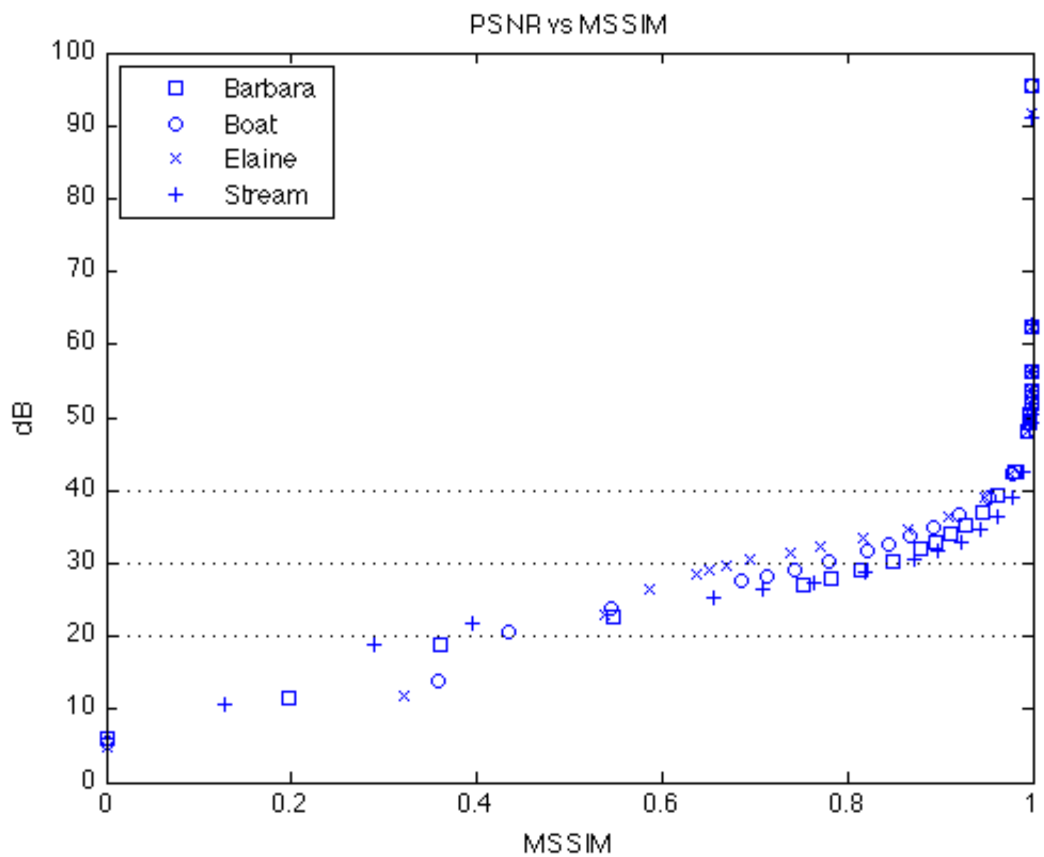
Imagery metrics: bpp vs PSNR



Imagery metrics: bpp vs MSSIM



Imagery metrics: PSNR vs MSSIM



Compression results

Original

Compressed

SSIM

$\varepsilon = 32$, $c = 4$

PSNR = 36.5220 dB, MSSIM = 0.9104, nsbr = 0.1609 bpp

Back to our original problem

k = 40 (62.5%)

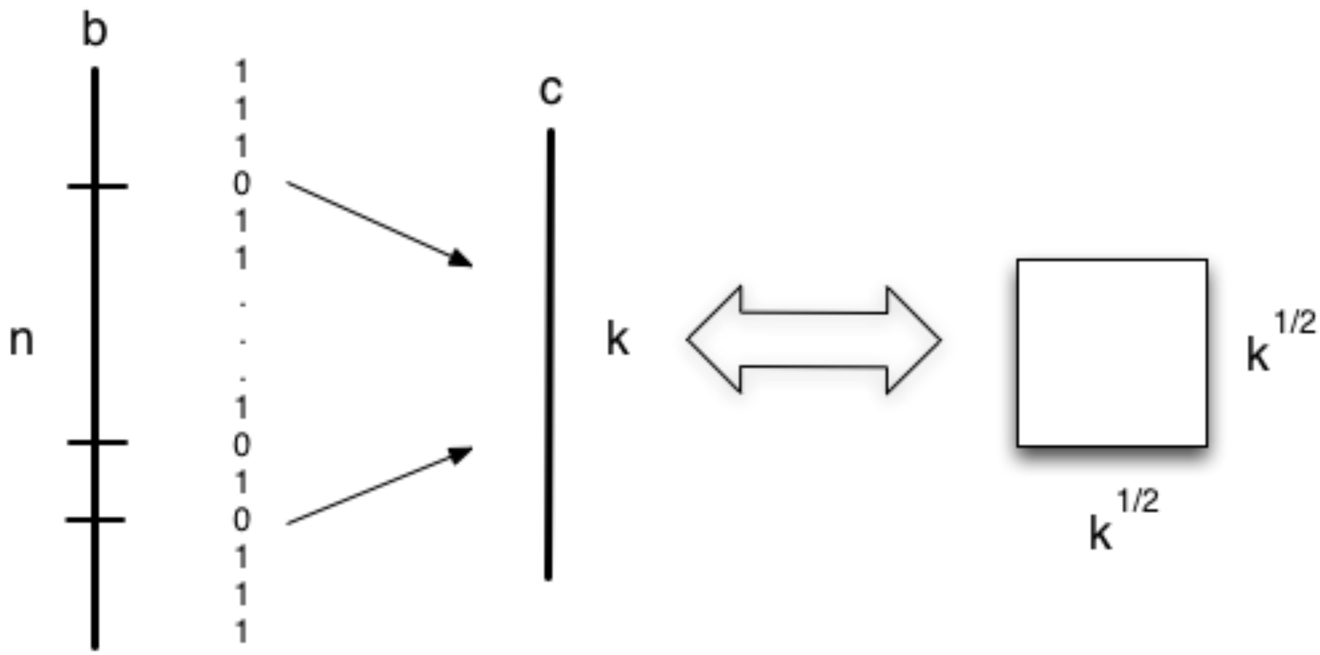
k = 32 (50%)

Compressed sensing and sampling

$$\min_{\mathbf{x}} \|\mathbf{x}\|_0 \text{ subject to } \|\mathbf{PA}\,\mathbf{x} - \mathbf{c}\,\|_2 < \varepsilon$$

$\mathbf{P}$ in $R^{k \times n}$, $\mathbf{A}$ in $R^{n \times m}$, and $\mathbf{c}$ in R^k

Deterministic sampling masks



$\varepsilon = c\sqrt{k}$, c = 4

Deterministic sampling masks

$\|\mathbf{A}' \mathbf{x}' - \mathbf{c}\|_2 < \varepsilon$, with $\mathbf{x}' = \mathrm{OMP}(\mathbf{A}', \mathbf{c}, \varepsilon)$, and $\mathbf{x}'$ in R^m

Deterministic sampling masks

$||\mathbf{A}'\,\mathbf{x}' - \mathbf{c}\,||_2 < \varepsilon$, with $\mathbf{x}' = \mathrm{OMP}(\mathbf{A}',\mathbf{c},\varepsilon)$, and $\mathbf{x}'$ in R^m

☒ $= c_3^{-1}(\mathbf{A}\,\mathbf{x}')$

Results



k = 40, c = 4

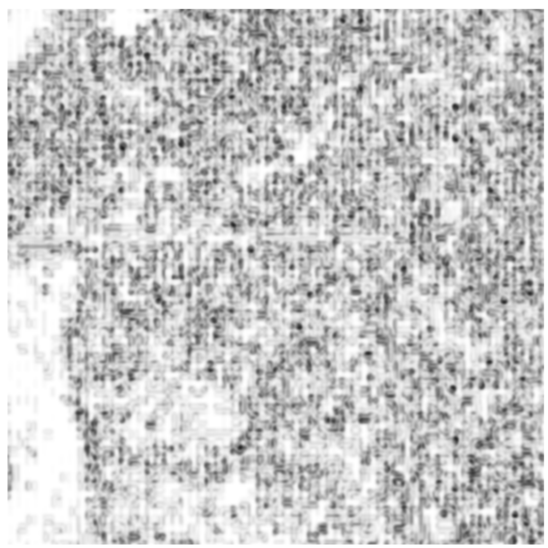
Luminance SSIM

k = 40, c = 4

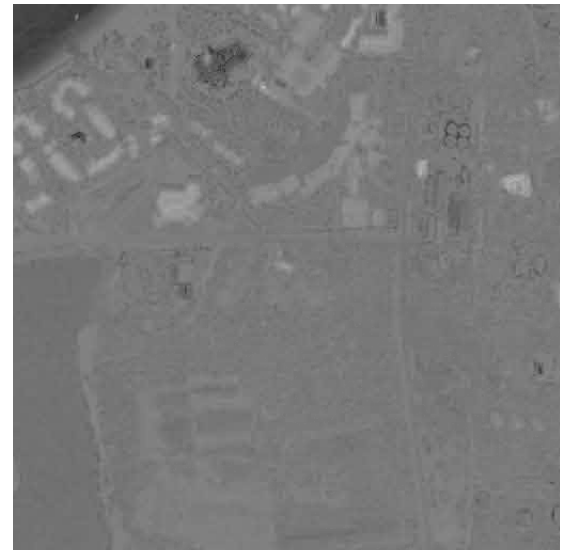
Results

PSNR = 21.1575

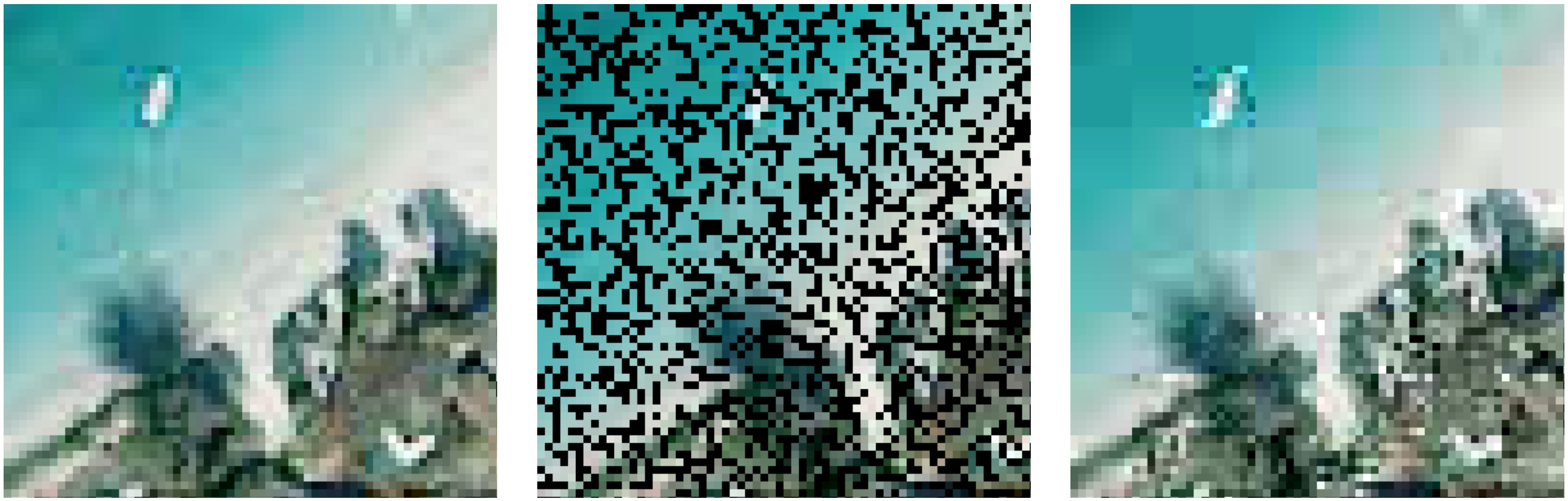
PSNR = 39.7019



PSNR = 39.4193



Results



k = 40, c = 4

Deterministic sampling masks
~ Inpainting?

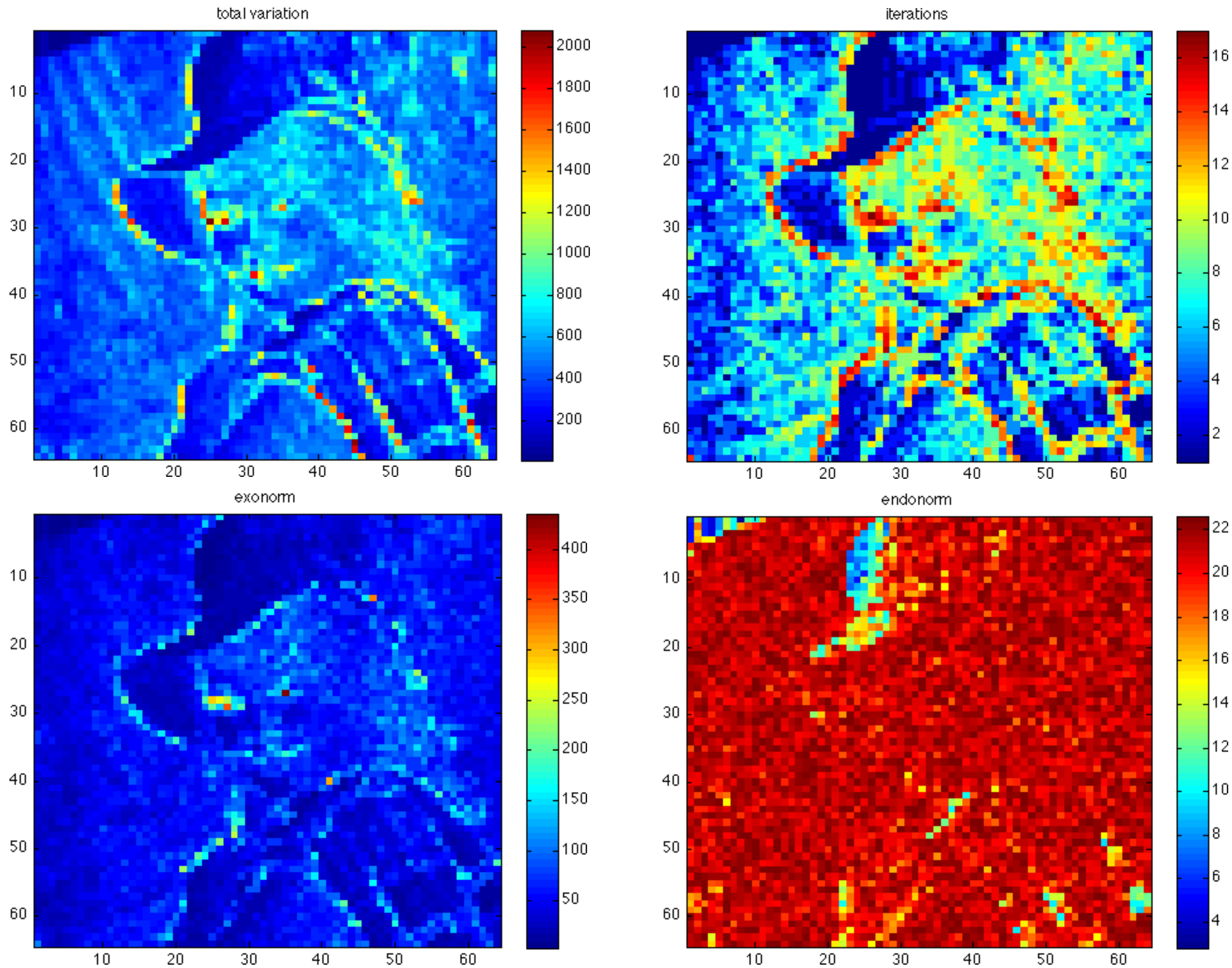
Results

k = 32, c = 4

PSNR = 29.8081 dB
MSSIM = 0.7461

Results

k = 32, c = 4



Thank you!

References

[1] A. M. Bruckstein, D. L. Donoho, and M. Elad, *From sparse solutions of systems of equations to sparse modeling of signals and images*, SIAM Review, 51 (2009), pp. 34–81.

[2] B. K. Natarajan, *Sparse approximate solutions to linear systems*, SIAM Journal on Computing, 24 (1995), pp. 227-234.

[3] G. W. Stewart, **Introduction to Matrix Computations**, Academic Press, 1973.

[4] T. Strohmer and R. W. Heath, *Grassmanian frames with applications to coding and communication*, Appl. Comput. Harmon. Anal., 14 (2004), pp. 257-275.

[5] D. S. Taubman and M. W. Mercellin, **JPEG 2000: Image Compression Fundamentals, Standards and Practice**, Kluwer Academic Publishers, 2001.

[6] G. K. Wallace, *The JPEG still picture compression standard*, Communications of the ACM, 34 (1991), pp. 30-44.

[7] S. Mallat, **A Wavelet Tour of Signal Processing**, Academic Press, 1998.

[8] Z. Wang, A.C. Bovik, H.R. Sheikh and E.P. Simoncelli, *Image quality assessment: from error visibility to structural similarity*, IEEE Transactions on Image Processing , vol.13, no.4 pp. 600- 612, April 2004.
https://ece.uwaterloo.ca/~z70wang/research/ssim/index.html