Deterministic sampling masks and compressed sensing: Compensating for partial image loss at the pixel level

Alfredo Nava-Tudela

Institute for Physical Science and Technology and Norbert Wiener Center, University of Maryland, College Park

Joint work with John J. Benedetto

Department of Mathematics and Norbert Wiener Center, University of Maryland, College Park

Overview

- Problem statement
- Image representation concepts
- Image compression basics
- Sparsity is the key, l_0-minimization, OMP
- Image compression revisited
- Imagery metrics
- Solving our problem: *deterministic sampling masks* and compressed sensing
- Solving our problem: results

Problem statement

Image: Satellite Imaging Corporation
http://www.satimagingcorp.com

Antigua

Problem statement

JPEG, JPEG 2000

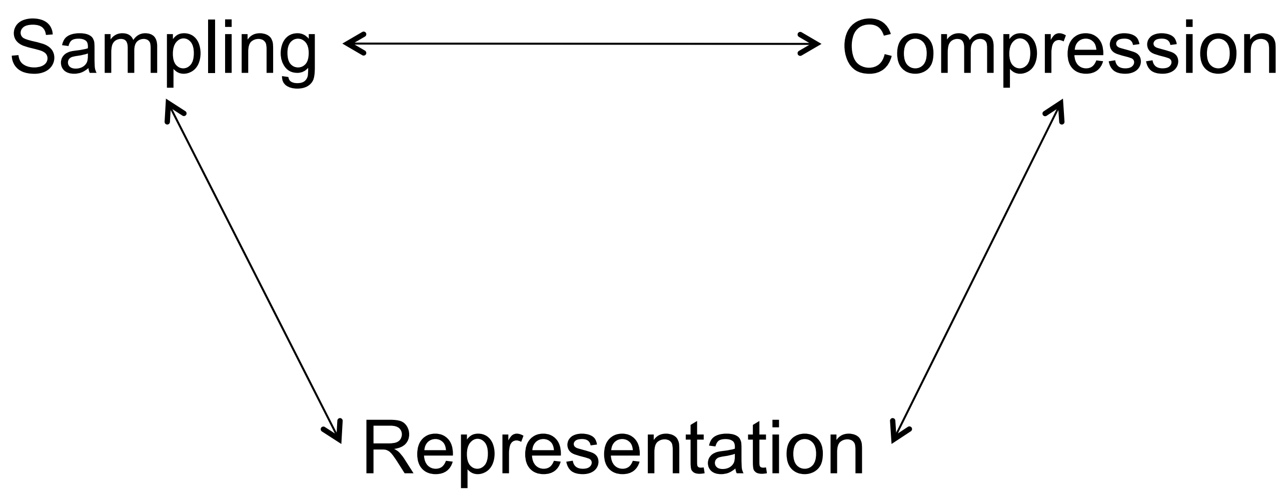


Image representation concepts[1]

Image representation concepts[1]

An image is…

Pixel = $I[n_1,n_2]$ = intensity, brightness, at $[n_1,n_2]$

n_2

n_1

$0 \le n_1 < N_1, \; 0 \le n_2 < N_2$

Image representation concepts[1]

$I[n_1,n_2] \in \{0, \ldots, 2^B-1\}$, or

$I[n_1,n_2] \in \{-2^{B-1}, \ldots, 2^{B-1}-1\}$, where

$I[n_1,n_2] = \text{round}(2^B\ J[n_1,n_2])$ and
$J[n_1,n_2] \in [0,1)$ or $[-½, ½)$

B is the depth of the image

Image representation concepts[1]

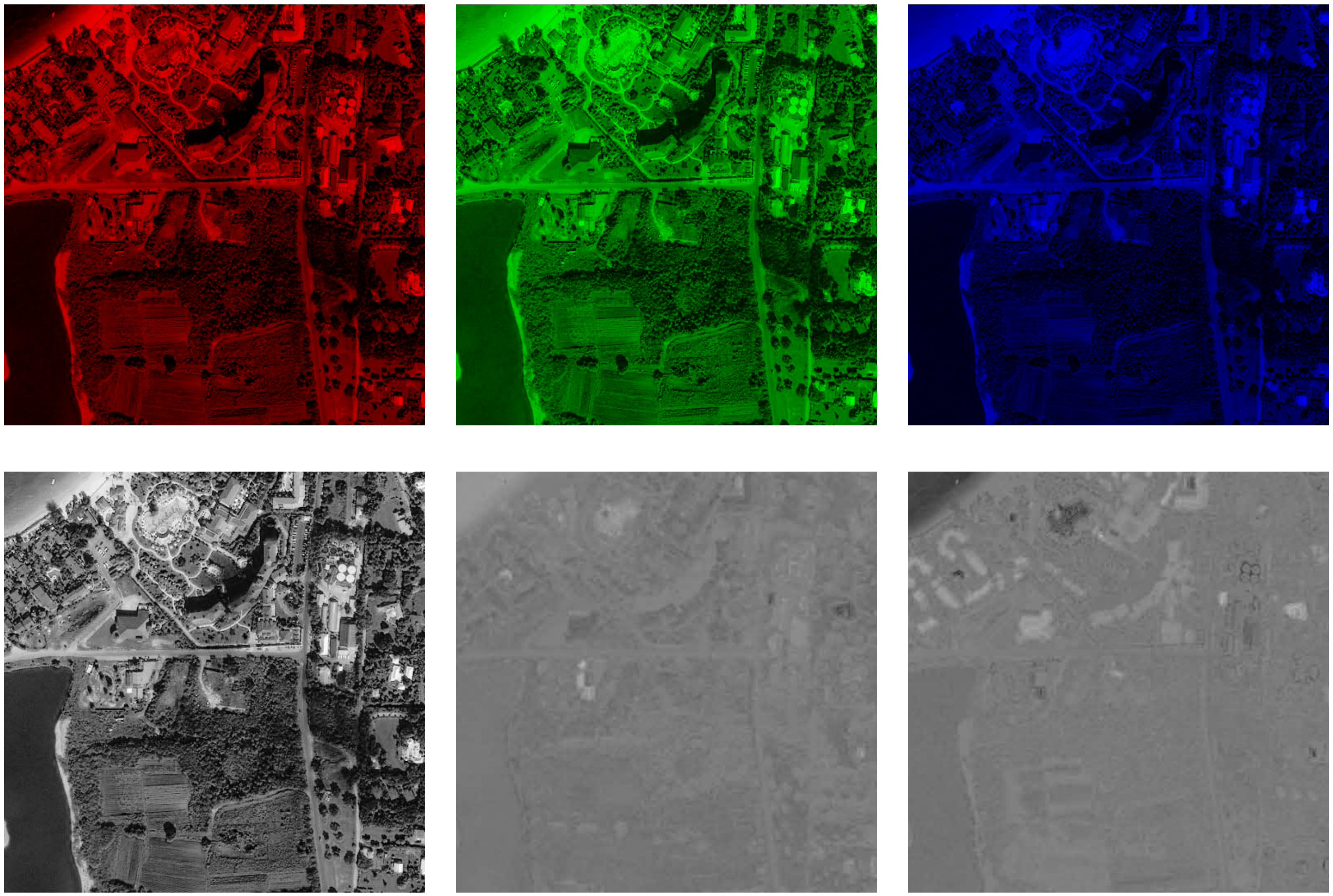


Image compression[1]

$$512 \times 512 \times 8 \times 3 = 6{,}291{,}456 \text{ bits}$$

Image compression[1]

JPEG, JPEG 2000

Image compression[1]

1) Partitioning of the image I in sub-images

[1] D. S. Taubman and M. W. Mercellin, **JPEG 2000: Image Compression Fundamentals, Standards and Practice**, Kluwer Academic Publishers, 2001.

Image compression[1]

1) Partitioning of the image I in sub-images
2) Transform sub-images to exploit correlations within them

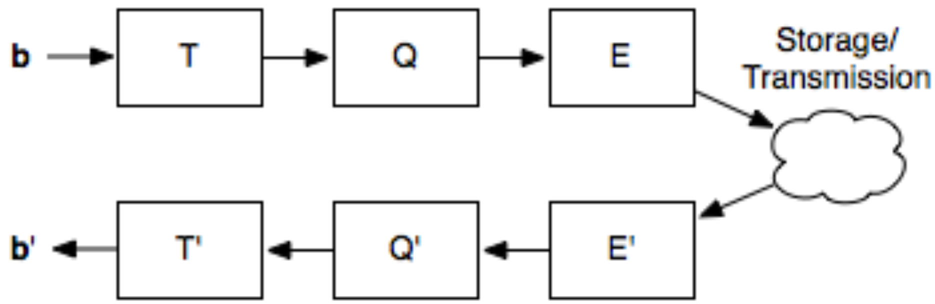
[1] D. S. Taubman and M. W. Mercellin, **JPEG 2000: Image Compression Fundamentals, Standards and Practice**, Kluwer Academic Publishers, 2001.

Image compression[1]

1) Partitioning of the image I in sub-images
2) Transform sub-images to exploit correlations within them
3) Quantize and encode

[1] D. S. Taubman and M. W. Mercellin, **JPEG 2000: Image Compression Fundamentals, Standards and Practice**, Kluwer Academic Publishers, 2001.

Image compression



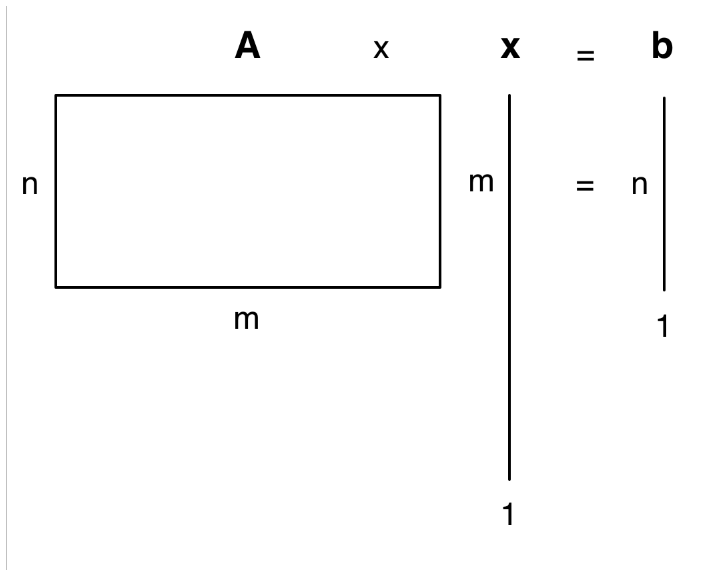
Sparsity is the key

Cn u rd ths?

vs

Can you read this?

Sparsity is the key



Sparsity

The l_0 "norm":

$$||\mathbf{x}||_0 = \#\ \{k : x_k \neq 0\}$$

l_0-minimization ~ sparse solution

$$(P_0): \quad \min_{\mathbf{x}} ||\mathbf{x}||_0 \text{ subject to } ||\mathbf{Ax} - \mathbf{b}||_2 = 0$$

l_0-minimization ~ sparse solution

(P_0^{ε}): $\min_{\mathbf{x}} ||\mathbf{x}||_0$ subject to $||\mathbf{A}\mathbf{x} - \mathbf{b}||_2 < \varepsilon$

l_0-minimization ~ sparse solution

(P_0^{ε}): $\min_{\mathbf{x}} ||\mathbf{x}||_0$ subject to $||\mathbf{A}\mathbf{x} - \mathbf{b}||_2 < \varepsilon$

Solving (P_0^{ε}) is NP-hard![2]

Is there any hope?

[2] B. K. Natarajan, *Sparse approximate solutions to linear systems*, SIAM Journal on Computing, 24 (1995), pp. 227-234.

Finding sparse solutions:OMP

Orthogonal Matching Pursuit algorithm: [3]

Task: Approximate the solution of $(P_0) : \min_{\mathbf{x}} ||\mathbf{x}||_0$ subject to $\mathbf{Ax} = \mathbf{b}$.

Parameters: We are given the matrix $\mathbf{A}$, the vector $\mathbf{b}$, and the threshold ϵ_0.

Initialization: Initialize $k = 0$, and set

- The initial solution $\mathbf{x}^0 = \mathbf{0}$.
- The initial residual $\mathbf{r}^0 = \mathbf{b} - \mathbf{Ax}^0 = \mathbf{b}$.
- The initial solution support $\mathcal{S}^0 = Support\{\mathbf{x}^0\} = \emptyset$.

Main Iteration: Increment k by 1 and perform the following steps:

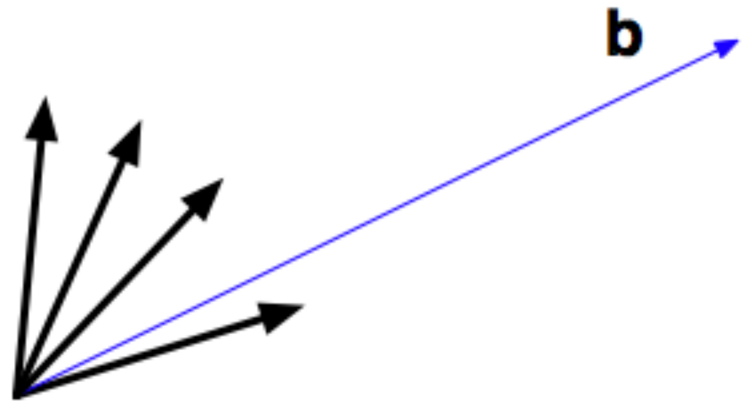
- **Sweep:** Compute the errors $\epsilon(j) = \min_{z_j} ||z_j \mathbf{a}_j - \mathbf{r}^{k-1}||_2^2$ for all j using the optimal choice $z_j^* = \mathbf{a}_j^T \mathbf{r}^{k-1} / ||\mathbf{a}_j||_2^2$.
- **Update Support:** Find a minimizer j_0 of $\epsilon(j)$: $\forall j \notin \mathcal{S}^{k-1}$, $\epsilon(j_0) \leq \epsilon(j)$, and update $\mathcal{S}^k = \mathcal{S}^{k-1} \cup \{j_0\}$.
- **Update Provisional Solution:** Compute $\mathbf{x}^k$, the minimizer of $||\mathbf{Ax} - \mathbf{b}||_2^2$ subject to $Support\{\mathbf{x}\} = \mathcal{S}^k$.
- **Update Residual:** Compute $\mathbf{r}^k = \mathbf{b} - \mathbf{Ax}^k$.
- **Stopping Rule:** If $||\mathbf{r}^k||_2 < \epsilon_0$, stop. Otherwise, apply another iteration.

Output: The proposed solution is $\mathbf{x}^k$ obtained after k iterations.

[3] A. M. Bruckstein, D. L. Donoho, and M. Elad, *From sparse solutions of systems of equations to sparse modeling of signals and images*, SIAM Review, 51 (2009), pp. 34–81.

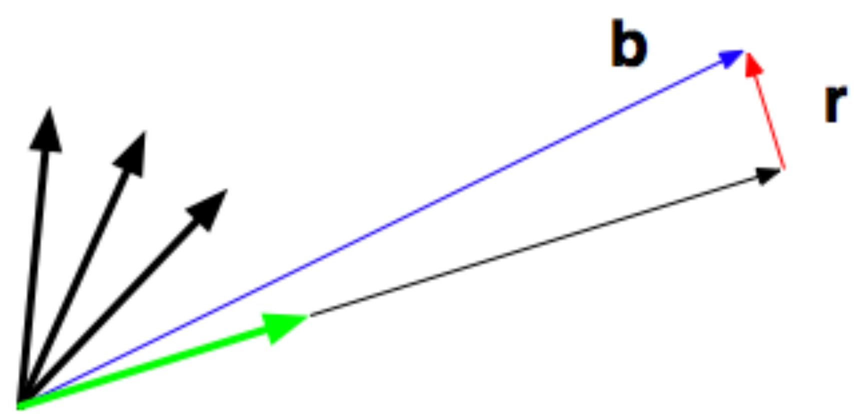
Finding sparse solutions:OMP

Orthogonal Matching Pursuit algorithm:



Finding sparse solutions:OMP

Orthogonal Matching Pursuit algorithm:



Finding sparse solutions:OMP

Orthogonal Matching Pursuit algorithm:

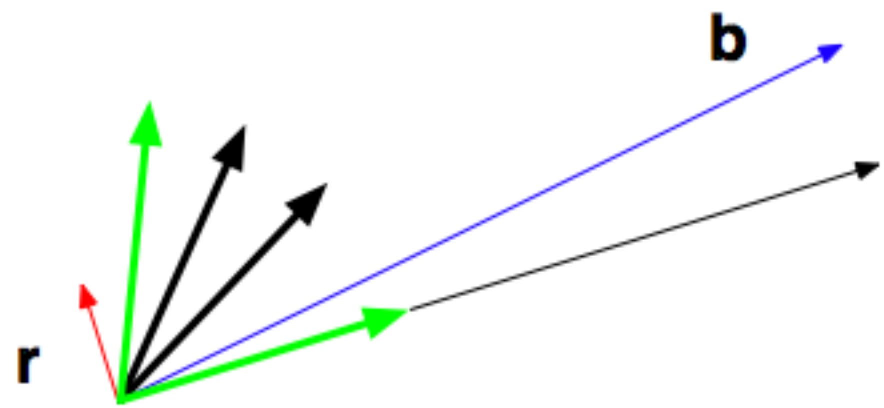
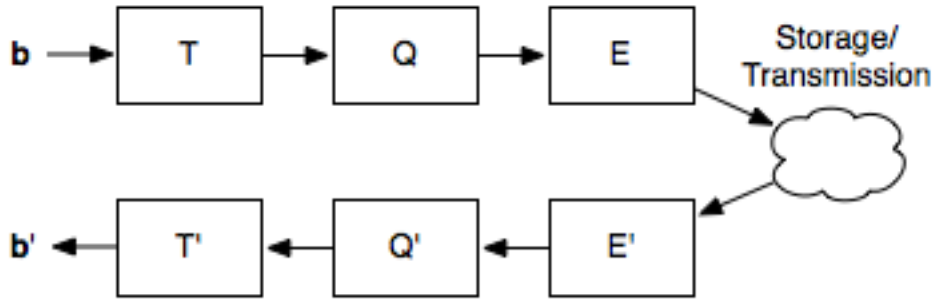
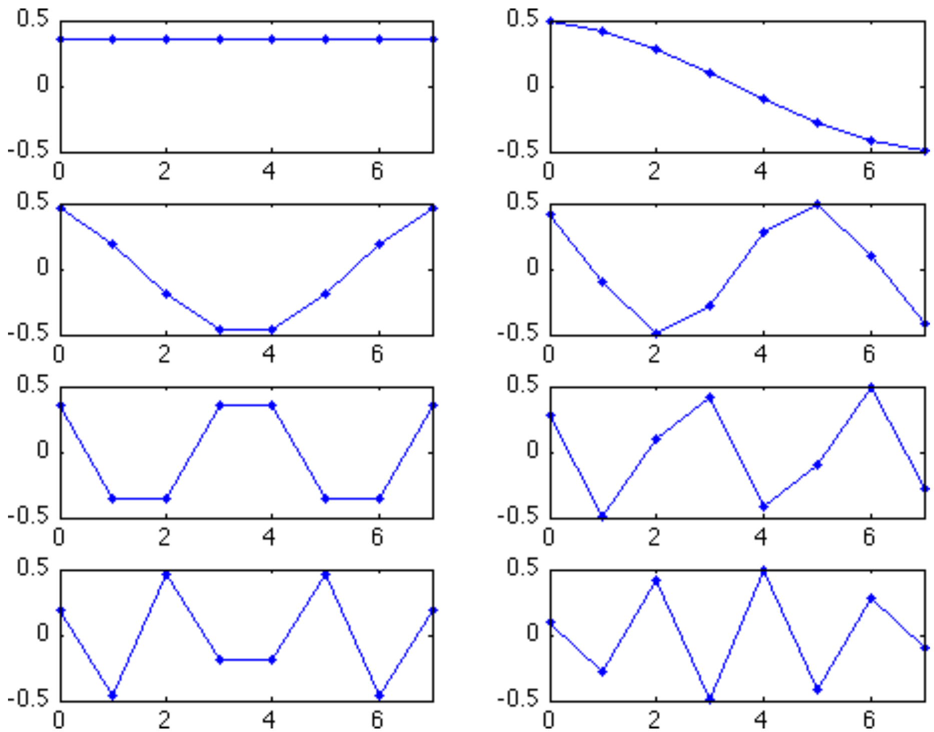


Image compression

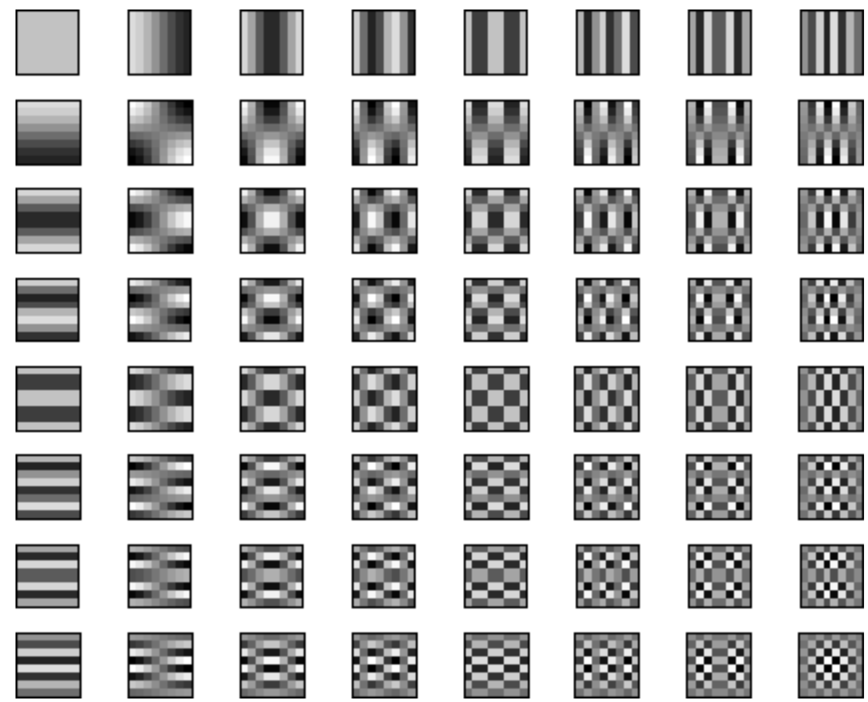


$\mathbf{T} = \mathbf{T}_{\varepsilon} = \mathrm{OMP}(\mathbf{A}, -, \varepsilon)$, $\mathbf{T}' = \mathbf{A}$

We need a matrix **A**

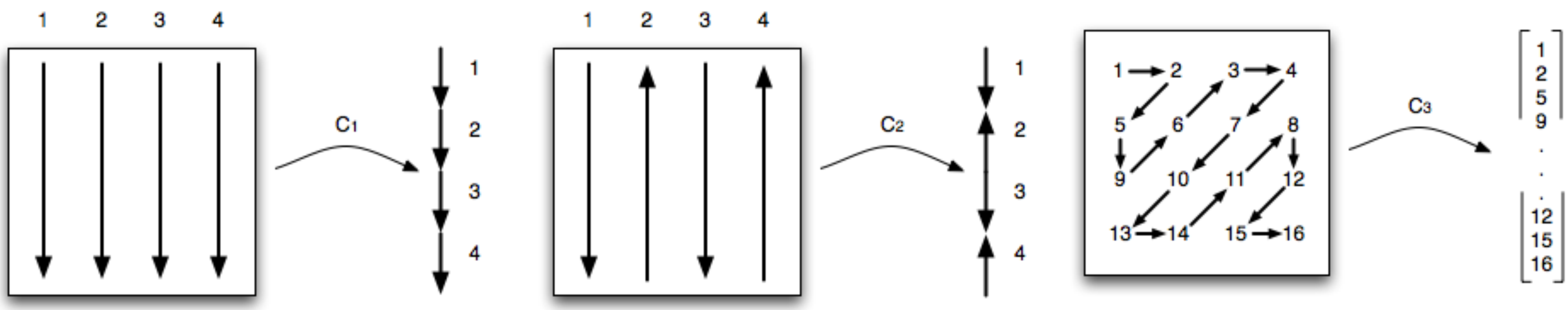


DCT

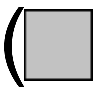
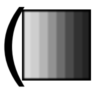
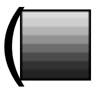
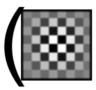


DCT2

We need a matrix **A**

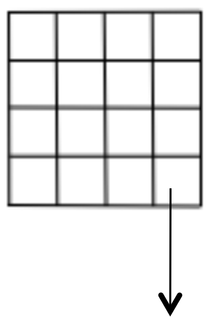


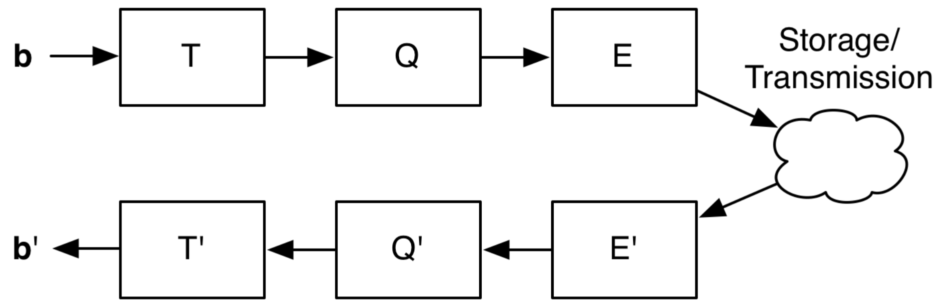
We need a matrix **A**

A = (c_3() c_3() c_3() … c_3())

Compressing a test image

Image





$c_3(\square) = \mathbf{b}$

$\mathbf{x}_0 = \mathbf{T}\ \mathbf{b} = \mathrm{OMP}(\mathbf{A}, \mathbf{b}, \varepsilon)$

$\boxtimes = c_3^{-1}(\mathbf{b}')$

$\mathbf{b}' = \mathbf{T}'\ \mathbf{x}_0 = \mathbf{A}\ \mathbf{x}_0$

Compressing a test image

☐ ~ ☒ ?　　　$\| \mathbf{b} - \mathbf{b}' \|_2 < \varepsilon$

But what does that mean visually?
How many bits were used?

Imagery metrics

Peak Signal-to-Noise Ratio (PSNR), measured in dB:

$$\mathrm{PSNR}(\mathbf{X},\mathbf{Y}) = 20 \log_{10}(\mathrm{MAX}_B / \sqrt{\mathrm{MSE}}),$$

with $\mathrm{MAX}_B = 2^B-1$, and $\mathrm{MSE} = \sum_{i,j} [\mathbf{X}(i,j) - \mathbf{Y}(i,j)]^2 / nm.$
In our case, $n = m = 512$, and B = 8, i.e. $\mathrm{MAX}_B = 255$.

Imagery metrics

Structural Similarity (SSIM), and Mean Structural Similarity(MSSIM) indices: [4]

$$\mathrm{SSIM}(\mathbf{x},\mathbf{y}) = \frac{(2\,\mu_x\,\mu_y + C_1)\,(2\,\sigma_{xy} + C_2)}{(\mu_x^2 + \mu_y^2 + C_1)\,(\sigma_x^2 + \sigma_y^2 + C_2)}$$

$$\mathrm{MSSIM}(\mathbf{X},\mathbf{Y}) = \frac{1}{M}\sum_{j=1}^{M}\mathrm{SSIM}(\mathbf{x}_j,\mathbf{y}_j)$$

[4] Z. Wang, A.C. Bovik, H.R. Sheikh and E.P. Simoncelli, *Image quality assessment: from error visibility to structural similarity*, IEEE Transactions on Image Processing, vol.13, no.4 pp. 600- 612, April 2004.

Imagery metrics

The normalized sparse bit-rate is

$$\mathrm{nsbr}(I,\mathbf{A},\varepsilon) = \sum ||\mathbf{x}_j||_0/N_1N_2,$$

where image I is of size N_1 by N_2.

Compression results

Original

Compressed

SSIM

$\varepsilon = 32 \Leftrightarrow d = 4$, average error per pixel for 8 x 8 blocks

PSNR = 36.6427 dB, MSSIM = 0.9767, nsbr = 0.3904 bpp

Back to our original problem

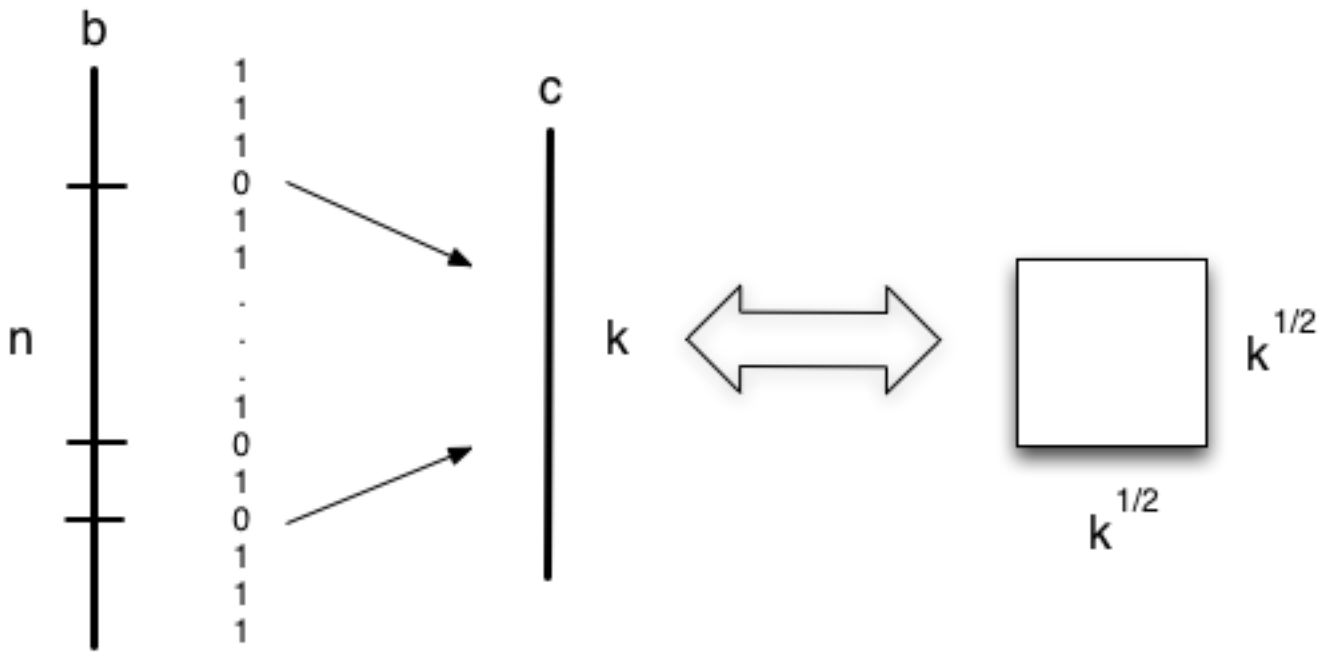
k = 40 (62.5%)

Compressed sensing and sampling

$$\min_{\mathbf{x}} ||\mathbf{x}||_0 \text{ subject to } ||\mathbf{PA}\,\mathbf{x} - \mathbf{c}\,||_2 < \varepsilon$$

$\mathbf{P}$ in $R^{k \times n}$, $\mathbf{A}$ in $R^{n \times m}$, and $\mathbf{c}$ in R^k

Deterministic sampling masks



If d = 4, then use ε = d $\sqrt{k}$

Deterministic sampling masks

$||\mathbf{A}'\,\mathbf{x}' - \mathbf{c}\,||_2 < \varepsilon$, with $\mathbf{x}' = \mathrm{OMP}(\mathbf{A}', \mathbf{c}, \varepsilon)$, and $\mathbf{x}'$ in R^m

Deterministic sampling masks

$$||\mathbf{A}' \mathbf{x}' - \mathbf{c}\,||_2 < \varepsilon, \text{ with } \mathbf{x}' = \mathrm{OMP}(\mathbf{A}', \mathbf{c}, \varepsilon), \text{ and } \mathbf{x}' \text{ in } R^m$$

$$\boxtimes = c_3^{-1}(\mathbf{A}\,\mathbf{x}')$$

Results

Original

Masked

Reconstruction

$k = 40$ (62.5%)
$d = 4$

Luminance SSIM

Luminance:
PSNR = 21.2002 dB
MSSIM = 0.7577

C_B:
PSNR = 39.7391 dB
MSSIM = 0.9357
C_R:
PSNR = 39.4362 dB
MSSIM = 0.9345

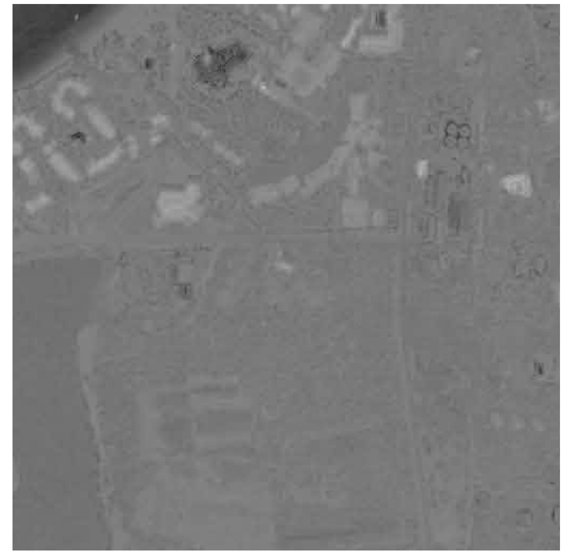
Results

k = 40 (62.5%)
d = 4

PSNR = 21.2002

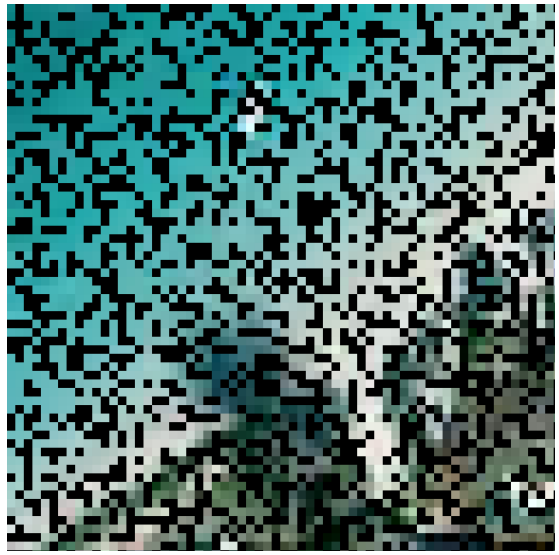
PSNR = 39.7391

PSNR = 39.4362

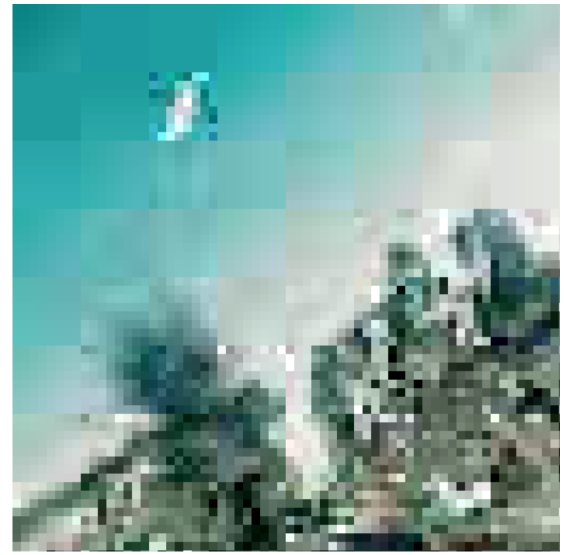


Results

Original detail



Masked detail



Reconstruction detail

k = 40 (62.5%)
d = 4

Deterministic sampling masks
~ In-painting?

Thank you!