Introduction to Compressive Sensing

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Initial Example
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- Picture has 10 megapixels
  - Effectively 10 MB of information
- Can store picture as less than 1 MB using .jpg
  - Image is compressible in wavelet basis
- Camera observes in elementary basis

Questions

1) Why was it necessary to collect all 10 MB of information, but throw 9 MB away?
2) Can we measure in a different basis?
Answer: For sparse objects, observe randomly in different basis.

Method

- $f \in \mathbb{C}^N$ is original signal, $\hat{f} \in \mathbb{C}^N$ is Fourier Transform
- Observe small number of random Fourier coefficients $\hat{f}(\gamma)$
- Wish to find sparsest solution $g \in \mathbb{C}^N$ such that
  \[
  \hat{g}(\gamma) = \hat{f}(\gamma), \quad \forall \gamma \text{ randomly observed}
  \]
- Compressive Sensing claims that sparsest $g$ is equal to $f$
Overview of the Problem

- $f \in \mathbb{C}^N$ be sparse, and choose some $\Omega \subset \mathbb{Z}_N$
- Let your measurements be $y$, where
  \[ y = \hat{f}|_\Omega \]
- Can recover “sparsest” solution by solving
  \[ \min_{g \in \mathbb{C}^N} \|g\|_{L^0(\mathbb{Z}_N)}, \quad \hat{g}|_\Omega = \hat{f}|_\Omega \]

Definition ($L^0$ norm)

\[ \|f\|_{L^0(\mathbb{Z}_N)} = |\{x \in \mathbb{Z}_N : f[x] \neq 0\}| \]
Overview of the Problem

- $\| \cdot \|_0$ is not computationally efficient
  - Non-convex problem
  - NP-Hard

Main Questions

1. Is there a metric other than $\| \cdot \|_0$ minimization?
2. How do we define “sparse”?
3. What is the minimum size of $\Omega$ needed?
Outline

1. Introduction
2. Compressive Sensing in Different Basis
3. Applications to Medical Imaging
4. Applications to Background Subtraction
5. Conclusion
Candés, Romberg, and Tao proposed:

\[ \min_{g \in \mathbb{C}^N} \|g\|_{L^1}, \quad \hat{g}|_{\Omega} = \hat{f}|_{\Omega}, \]  

L^1 Minimization

Problem can be solved using Interior Points Method
  - Modified Newton’s method
Remember: Fourier coefficients are sampled randomly
  - If desire \( m \) samples, choose \( \Omega \) uniformly at random over all \( |\Omega| = m \)
Theorem (Candés, Romberg, Tao)

Let $f \in \mathbb{C}^N$ be some discrete signal with support set $T$, where $T$ is unknown. Choose $\Omega$ of size $|\Omega| = m$ uniformly at random. For a given accuracy parameter $M$, if

$$|T| \leq C_M (\log N)^{-1} |\Omega|,$$

(2)

then with probability exceeding $1 - O\left( N^{-M} \right)$, the minimizer to problem (1) is unique and equal to $f$.

$$C_M \sim O\left( \frac{1}{M} \right)$$
Generalize to $N \times N$ orthogonal matrix $U$ such that $U^* U = N \cdot I_N$.

Observe $y = U_\Omega f$.

Recover sparse $f \in \mathbb{C}^N$ by solving

$$\min_{g \in \mathbb{C}^N} \|g\|_{L^1}, \quad U_\Omega g = U_\Omega f.$$ (3)

Successful with high probability given

$$|\Omega| \geq C \cdot [\mu(U)]^2 \cdot |T| \cdot \log(N)$$

where

$$\mu(U) = \max_{k,j} |U_{k,j}|$$
Main Theorem

Theorem (Candés, Romberg)

Fix $T \subset \mathbb{Z}_N$. Let $U$ be an $N \times N$ orthogonal matrix with $\mu = \max_{i,j} |U_{i,j}|$.

Choose a sign sequence $z(t)$ for $t \in T$, uniformly at random. Choose $\Omega$ at random such that

$$|\Omega| \geq C_0 |T| \mu^2(U) \log(N/\delta) \quad \text{and} \quad |\Omega| \geq C'_0 \log^2(N/\delta)$$

Let $f \in \mathbb{C}^N$ have $\text{supp}(f) = T$ and

$$\text{sgn}(f)(t) = z(t), \quad \forall t \in T.$$

Then with probability exceeding $1 - \delta$, $f$ is the unique minimizer to (3).
Alternate Interpretation of $U$

Consider $U = \Phi \Psi$

- $\Psi$ is sparsity basis, $\Psi^* \Psi = I$
- Call $\Phi$ measurement basis, $\Phi^* \Phi = N \cdot I$

Corollary (Sparsity and Measurement Basis)

Let $x \in \mathbb{C}^N$ (not necessarily sparse). Wish to recover $x$ from

$$y = \Phi_\Omega x.$$ 

Assume $\exists$ sparse $f$ such that $x = \Psi f$, so

$$y = \Phi_\Omega \Psi \cdot f = U_\Omega f.$$ 

If $f^\#$ minimizes (3), best estimate for $x$ is

$$x^\# = \Psi f^\#.$$
Consider only observing rank $r$ matrix $M \in \mathbb{C}^{N \times N}$ on some subset $\Omega$ of its indices.

Let SVD of $M$ be $M = U\Sigma V^T$.

Possible to recover $M$ as solution to

$$\min_X \text{rank}(X) \quad \text{such that} \quad (UXV^T)_{i,j} = (U\Sigma V^T)_{i,j}, \quad (i,j) \in \Omega$$

This problem is NP-hard.
Matrix Completion Approach

**Definition (Nuclear Norm)**

Let $\sigma_i(M)$ be the $i^{th}$ largest singular value of $M$. If $\text{rank}(M) = r$, then

$$\|M\|_* := \sum_{i=1}^{r} \sigma_i(M)$$

**Recovery Algorithm**

Wish to recover $M$ by solving problem $(P^*)$, which is

$$\min_X \|X\|_*$$

such that $(UXV')_{i,j} = M_{i,j}, \ (i,j) \in \Omega$
Applications to Nuclear Magnetic Resonance

- Nuclear Magnetic Resonance (NMR) imaging studies molecular structure.
- Multidimensional correlations found can identify and study fluid-saturated porous medium
  - Specifically can use T1-T2 relaxation times (longitudinal and traverse)
  - Collisions of spin-bearing molecules with pore walls induce more rapid relaxation
  - Simple correlation between relaxation rate and pore surface-to-volume ratio
- Best way to measure T1-T2 times is using pulse train of RF energy particles
- Problem is NMR is incredibly slow
Math Behind NMR

- Echo measurements are related to T1-T2 correlations via Laplace Transform

\[ M(\tau_1, \tau_2) = \int \int (1 - 2e^{\tau_1/T_1})e^{\tau_2/T_2} F(T_1, T_2) dT_1 dT_2 + E(\tau_1, \tau_2) \]

- We’ll consider more general 2D Fredholm Integral

\[ M(\tau_1, \tau_2) = \int \int k_1(\tau_1, T_1)k_2(\tau_2, T_2) F(T_1, T_2) dT_1 dT_2 + E(\tau_1, \tau_2) \]

where \( E(\tau_1, \tau_2) \sim \mathcal{N}(0, \epsilon) \)

- Discretize to

\[ M = K_1 F K'_2 + E \]
Recovery from Small Number of Entries
Error Analysis

Difference Between M and Recovery
Video Presentation
Robust PCA

Principal Component Pursuit

Let $L_0$ be low rank background and $S_0$ be sparse foreground. Wish to recover $L_0$ and $S_0$ by solving

$$\min_{L,S} \|L\|_* + \lambda \|S\|_1$$

such that $L + S = M$
Video Split
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**Final Thoughts**

- Whole field based around observations being redundant
- In reality, most objects can be represented more sparsely in different way
- Still large number of applications that can benefit
- (Wojtek made me put this in) NWC has many more problems of interest