Exploiting Data-Dependent Structure for Improving Sensor Acquisition and Integration

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Outline

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Data-Dependent Structure

- Advancements in sensor construction and production cost has led to a deluge of data
- Thesis utilizes data-dependent operators to discover efficient representations of data
- This focus on learning structure splits into three topics
  1. Building data-dependent graphs to capture structure and detect anomalous objects
  2. Fusing low-dimensional parameters from heterogeneous data sources
  3. Exploiting compressibility of data to reduce sampling requirements prior to collection

Reduced Acquisition Time

- Based on the theory of compressive sensing and matrix completion
  - Recover signal that is *sparse* in some basis
  - Key is that measurements are randomly made and *incoherent* with respect to sparsity basis
  - Utilizes convex relaxation and optimization schemes to reconstruct signal
  - Reconstruction only requires $\mathrm{O}(K \log N)$ measurements

Contributions of Thesis

- Proved *bounded norm Parseval frames* satisfy necessary conditions for matrix reconstruction
- Demonstrated use of matrix completion for solving 2D Fredholm integrals from incomplete measurements
- Improved acquisition time for nuclear magnetic resonance spectroscopy via reducing necessary number of samples

Fusing Low Dimensional Parameters of High Dimensional Data

- Based on graph and operator theoretic approaches to pattern recognition and machine learning
  - Builds operator that encodes similarity between data points
  - Takes data from high-dimensional data space and embeds into low-dimensional euclidean space
  - Allows common comparison across heterogeneous sensors

Contributions of Thesis

- Built approximate inversion algorithm for Laplacian eigenmaps that utilizes compressive sensing
- Used inversion along with Coifman and Hirn's graph rotation to create data fusion algorithm
- Reconstructed missing LIDAR data (altitudes) from hyperspectral camera images (electromagnetic spectrum frequencies)

Laplacian Eigenmaps

- Let $\Omega = \{x_1, ...x_n\} \subset \mathbb{R}^d$ be a set of data points, or *data space*
- Idea is to learn structure via inter-data similarities
- Encode relationships via symmetric kernel $k : \Omega \times \Omega \to [0, 1]$
  - Gaussian kernel, $k(x, y) = e^{-\frac{\|x-y\|_2^2}{2\sigma^2}}$
  - Mahalanobis distance, $k(x, y) = e^{-(x-y)^T S^{-1}(x-y)}$
  - Graph adjacency, $k(x, y) = \begin{cases} 1 & : \quad x \in \mathcal{N}(y), \\ 0 & : \quad \text{otherwise.} \end{cases}$
- Build graph $G = (\Omega, E, W)$, where $\{x, y\} \in E \iff k(x, y) \approx 1$
  - $W_{x,y} = k(x, y)$ if $\{x, y\} \in E$
  - k-Nearest Neighbors
  - ϵ-Nearest Neighbors
- Key is that G is sparse

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Laplacian Eigenmaps (cont.)

- Calculate the normalized graph Laplacian $L = I - D^{-1/2} W D^{-1/2}$, where $D_{x,x} = \sum_y W_{x,y}$
- Solve the eigenvalue problem

$$L\phi_i = \lambda_i \phi_i$$

  - $0 = \lambda_0 \leq \lambda_1 \leq ... \leq \lambda_{n-1} \leq 2$
  - $\langle \phi_i, \phi_j \rangle = 0$ for $i \neq j$

LE Embedding

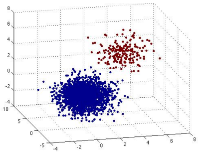
$$\begin{aligned} \Phi \quad &: \quad \Omega \to \mathbb{R}^m \\ x \quad &\mapsto \quad [\phi_1(x), ..., \phi_m(x)] \end{aligned}$$

- Forms low dimensional embedding that preserves local neighborhood structure
- Minimizes $\sum_{x,y} \|\Phi(x) - \Phi(y)\| \frac{W_{x,y}}{\sqrt{D_{x,x} D_{y,y}}}$

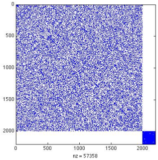
Graph Representation of Data Set

- Maps points from complicated data space to Euclidean feature space
  - $D_{LE}(x, y) = \|\Phi(x) - \Phi(y)\|_2$
- Can be used to reduce dimension of data

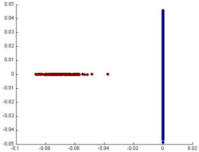
Original Data



Adjacency Matrix



LE Embedding



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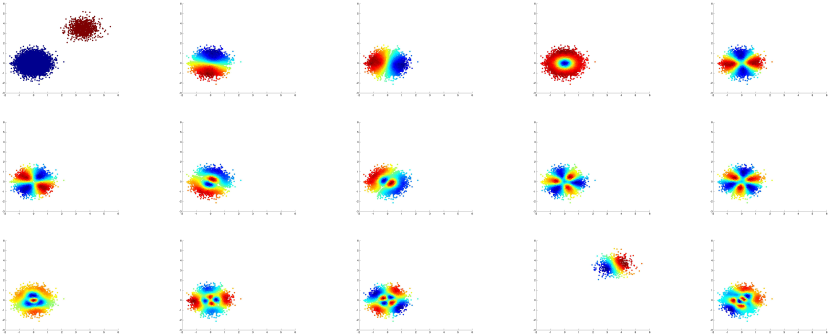
Complicated Distribution of Eigenvectors

- Most literature simply utilizes "first m eigenvectors with non-zero eigenvalue"
  - These correspond to "low frequency" information on graph
- When in doubt, simply be liberal with choice of m
- However, distribution of eigenvectors more complicated
  - Do not simply correspond to 1 eigenvector concentrated on each cluster
- Rest of talk is examination of eigenvector localization and order of emergence
  - Specifically when clusters are differing sizes

Examples of Small Clusters Failing to Emerge

Eigenvectors with non-zero eigenvalues



$|C_1| = 10,000, \ |C_2| = 1,000$

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Outline of Approach

- Assume graph $G = (\Omega, E)$ already formed from data, under some metric and using k-NN
- For simplicity, assume $\{x, y\} \in E \iff y \in \mathcal{N}(x)$ and $w_{x,y} = 1$
  - Approximates behavior of LE while utilizing vast literature on regular graph
- Wish to examine emergence of small clusters in eigenvectors
- Approach:
  1. Characterize eigenpairs of disjoint graphs with heterogeneous sized clusters
  2. Demonstrate that, upon adding edges to connect graph, eigenpairs do not deviate far from those of disjoint graph

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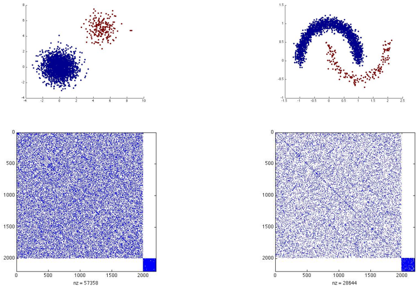
Similarity of Data Generated Graphs

- By analyzing graph, can bypass specifics of data set
- Characteristics such as convexity and scale can be ignored



Clusters as Regular Graphs

- Need way to characterize data clusters
- Define data cluster on n nodes to be *random regular graph*

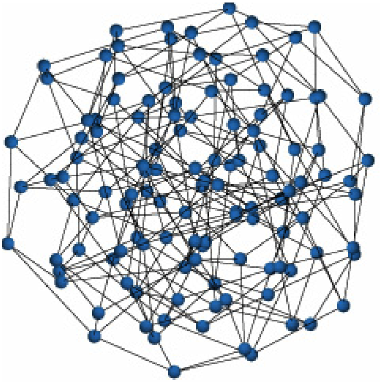
Definition (Family of Regular Graphs)

The family of regular graphs $\mathcal{G}_{n,k}$ is the set of all graphs $G = (V, E)$ such that:

1. V contains n nodes
2. $\forall x \in V, \deg(x) \equiv |\{y \in V : \{x, y\} \in E\}| = k$.

- Random regular graph is $G \in \mathcal{G}_{n,k}$ chosen uniformly at random from all graphs
- With high probability, G does not have large cycles or large complete subgraphs

Random Regular Graphs



Donetti, Neri, and Muño 2006

Validity of Regular Graph Assumption

- Ties into k-Nearest Neighbors edges for graph
- If ignoring need for weights to be symmetric, then exactly generates k-regular graph
- Following theory also applies for Erdös Renyi graph

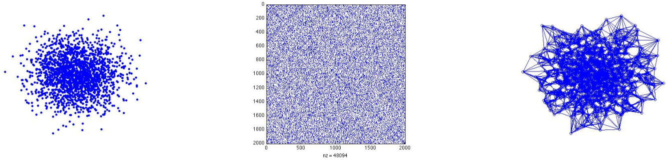
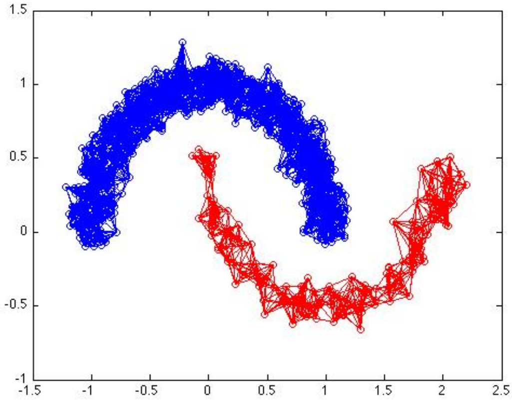


Figure: $\mu_{degree} = 24.05$, $\sigma_{degree} = 1.41$

Data Structure to Keep in Mind for Section



$$|C_1| = 2000, |C_2| = 200$$

Eigenvalues Determine Order of LE Feature Vectors

- Order in which LE eigenvectors appear determined by eigenvalue order
- Goal:
  - Characterize eigenvalues of two graph clusters separately
  - Examine interlacing of eigenvalues to determine order of features emerging
- Eigenvalue distribution of k-regular graph is well studied question
  - McKay, 1981 - showed empirical spectral distribution of $\frac{1}{\sqrt{k-1}}A_n$ converges to

$$f_{semi}(x) = \frac{1}{2\pi}\sqrt{4-x^2}, \quad -2 < x < 2$$

  - Dumitriu, Pal 2013 - found deviation from f_{semi} for finite graph
  - Independently found by Tran, Vu, and Wang 2013

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Eigenvalues of Regular Graph (cont.)

Theorem (Dumitriu, Pal, 2012)

Fix $\delta > 0$ and let $k = (\log(n))^{\gamma}$, and let $\eta = \frac{1}{2}(\exp(k^{-\alpha}) - \exp(-k^{-\alpha}))$, for $0 < \alpha < \min(1, 1/\gamma)$. Then for

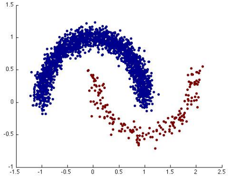
- *$G \in \mathcal{G}_{n,k}$ chosen randomly with adjacency matrix A, and*
- *interval $\mathcal{I} \subset \mathbb{R}$ such that $|\mathcal{I}| \geq \max\{2\eta, \eta/(-\delta \log \delta)\}$,*

there exists an N large enough such that $\forall n > N$,

$$|\mathcal{N}_{\mathcal{I}} - n \int_{\mathcal{I}} f_{semi}(x)dx| < n\delta|\mathcal{I}|$$

with probability at least $1 - o(1/n)$. Here, $\mathcal{N}_I$ is the number of eigenvalues of $\frac{1}{\sqrt{k-1}}A$ in the interval $\mathcal{I}$.

Eigenvalues for Disjoint Clusters



- Consider two clusters C_1 and C_2 with $|C_1| = D|C_2|$
  - G_1 and G_2 are generated graphs on C_1 and C_2, respectively
  - $G_1 \in \mathcal{G}_{n,k}$ and $G_2 \in \mathcal{G}_{\frac{n}{D},k}$
- $\sigma\left(\frac{1}{\sqrt{k-1}}A_1\right)$ and $\sigma\left(\frac{1}{\sqrt{k-1}}A_2\right)$ distributed similarly due to Dumitriu and Pal
  - $\sigma(L_1)$ and $\sigma(L_2)$ distributed similarly on $[0, 2]$
- Eigenvalues interweave in way that depends on D

Eigenvalues for Disjoint Clusters (cont.)

Theorem (C., 2014)

Let $G = (\Omega, E)$ be graph. Suppose Ω can be split into two disjoint regular graph clusters C_1 and C_2 such that $|C_1| = D|C_2| = n$. Choose any interval $\mathcal{I} \subset [0, 2]$ such that

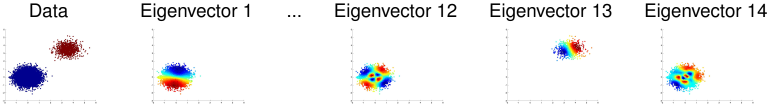
$$|\mathcal{I}| \geq \frac{\sqrt{k-1}}{k} \max\{2\eta, \eta/(-\delta \log \delta)\}.$$

Let L denote the graph Laplacian, with eigenpairs $\{(\sigma_i, v_i)\}_{i=1}^m$ that lie in $\mathcal{I}$. Let $\mathcal{N}_{\mathcal{I}}^1 = |\{i : \text{supp}(v_i) \subset C_1\}|$ and $\mathcal{N}_{\mathcal{I}}^2 = |\{i : \text{supp}(v_i) \subset C_2\}|$. Then $\mathcal{N}_{\mathcal{I}}^1 + \mathcal{N}_{\mathcal{I}}^2 = m$, and $\forall n > N$, with probability at least $1 - o(1/n)$,

$$|\mathcal{N}_{\mathcal{I}}^1 - D\mathcal{N}_{\mathcal{I}}^2| \leq 2\delta n \frac{k}{\sqrt{k-1}} |\mathcal{I}|.$$

Eigenvector Localization on Disjoint Graphs

- Theorem implies each eigenvector is localized on either C_1 or C_2
- Up to error, eigenvector on C_2 appears approximately $1 : D + 1$ times
  - Implies most of energy from LE embedding lies in C_1
- Applies for any interval $\mathcal{I} \subset [0, 2]$
- Can be generalized to larger number of clusters
- Argument explains initial example shown $(D = 10)$



Sketch of Proof for Disjoint Graphs

- $L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}$
  - $\sigma(L) = \sigma(L_1) \cup \sigma(L_2)$
  - $L \begin{pmatrix} v \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} v \\ 0 \end{pmatrix} \iff L_1 v = \lambda v$
  - $L \begin{pmatrix} 0 \\ v \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ v \end{pmatrix} \iff L_2 v = \lambda v$
- Thus all eigenvectors v of L concentrated on one cluster C_i
  - Order determined by $\sigma(L_1)$ and $\sigma(L_2)$
- Rescale $\sigma\left(\frac{1}{\sqrt{k-1}}A\right)$ from Dumitriu and Pal Theorem to $\sigma(L)$
  - Because G is k-regular,

$$\frac{1}{\sqrt{k-1}}Av_i = \lambda_i v_i \iff Lv_i = (1 - \frac{\sqrt{k-1}}{k}\lambda_i)v_i$$

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$$\frac{1}{\sqrt{k-1}} A v_i = \lambda_i v_i \iff L v_i = (1 - \frac{\sqrt{k-1}}{k} \lambda_i) v_i$$

Sketch of Proof (cont.)

- Design parameters from Dumitriu and Pal Theorem that are constant across both clusters C_i
  - Interval $\mathcal{I}$ for $\sigma(L)$ has corresponding interval $\mathcal{I}_A$ for $\sigma(\frac{1}{\sqrt{k-1}}A)$
- Theorem guarantees that

$$|\mathcal{N}^1_{\mathcal{I}_A} - n\int_{\mathcal{I}_A} f_d(x)dx| < n\delta|\mathcal{I}_A|,$$
$$|\mathcal{N}^2_{\mathcal{I}_A} - \frac{n}{D}\int_{\mathcal{I}_A} f_d(x)dx| < \frac{n}{D}\delta|\mathcal{I}_A|.$$

- This means

$$\begin{aligned}|\mathcal{N}^1_{\mathcal{I}} - D\mathcal{N}^2_{\mathcal{I}}| &\leq |\mathcal{N}^1_{\mathcal{I}_A} - n\int_{\mathcal{I}_A} f_d(x)dx| + |D\mathcal{N}^2_{\mathcal{I}_A} - n\int_{\mathcal{I}_A} f_d(x)dx| \\ &\leq 2n\delta|\mathcal{I}_A| \\ &= 2n\delta\frac{k}{\sqrt{k-1}}|\mathcal{I}|\end{aligned}$$

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- Design parameters from Dumitriu and Pal Theorem that are constant across both clusters C_i
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Disjoint Graph Conclusions

- Important notes from Theorem
  - Characterizes order of feature vectors from LE
  - Demonstrates that, among first m eigenvectors, $\frac{D}{D+1}$ of them are concentrated in largest cluster
  - Attempt to design LE similarity kernel such that graph as disjoint as possible
  - Arguments generalize to larger number of clusters
- Drawbacks
  - In practice, cannot design disconnected graph from data
  - Need to add edges to connect graph for better theory
  - Already know Fiedler vector is highly sensitive to connecting edge

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Weakly Connected Clusters

Definition

A graph with *weakly connected clusters of order t* is a connected graph with adjacency matrix

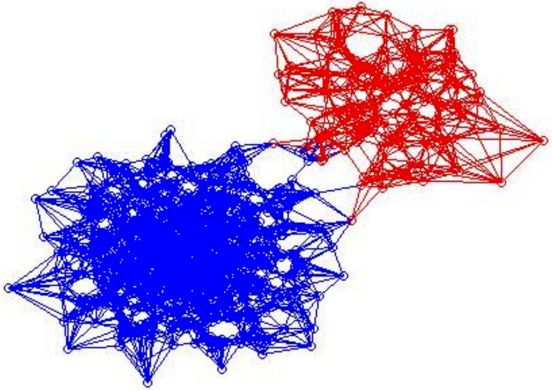
$$A = \begin{pmatrix} A_1 & B_{1,2} \\ B_{1,2}^T & A_2 \end{pmatrix},$$

where

1. A_1 and A_2 are adjacency matrices of k-regular graphs, and
2. $B_{1,2}$ has t non-zero entries.

We shall refer to the nodes of A_1 as C_1 and the nodes of A_2 as C_2

Weakly Connected Clusters Example



Weakly Connected Clusters as Matrix Perturbation

- Now problem is characterized by perturbation of known matrix
  - G is weakly connected graph with adjacency $A = \begin{pmatrix} A_1 & B_{1,2} \\ B_{1,2}^T & A_2 \end{pmatrix}$
  - H is disjoint graph with adjacency $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$
- Let L_G be normalized Laplacian of G, and similar for L_H

Perturbation of L_H

$L_G = L_H + E$, where $\|E\|_F \ll \|L_H\|_F$

- Questions:
  1. Is eigenvalue ordering of L_G drastically affected?
  2. Are eigenvectors of L_G still concentrated on clusters?

Eigenvalue Distribution for GWCC

Theorem (Chen, et. al., 2012)

Let $G = (\Omega, E_G)$ and $H = (\Omega, E_H)$ be spanning graphs such that $|E(G - H)| \leq t$. If

$$\lambda_1 \leq \ldots \leq \lambda_n, \text{ and } \theta_1 \leq \ldots \leq \theta_n$$

are the eigenvalues of the normalized Laplacians L_G and L_H respectively, then

$$\theta_{i-t} \leq \lambda_i \leq \theta_{i+t}, \qquad 1 \leq i \leq n,$$

with the convention that $\theta_{-t} = \ldots = \theta_0 = 0$ and $\theta_{n+1} = \ldots = \theta_{n+t} = 2$.

- Related to Weyl's inequality and Courant-Fischer theorem
- Shows why lowest eigenvalues difficult to predict
- Will lead to issues with Fiedler vector

Eigenvalue Distribution for GWCC (cont.)

Lemma (C., 2014)

Let $G = (\Omega, E)$ be a graph with weakly connected clusters of order t, with

- $|C_1| = n$,
- $|C_2| = \frac{n}{D}$.

Fix δ, k, α, η, and $\mathcal{I}$ as in Theorem for disjoint clusters. Let L denote the graph Laplacian, and $\sigma_1, \ldots, \sigma_m$ denote the m eigenvalues of L that lie in $\mathcal{I}$. Then m satisfies

$$|m - (n + \frac{n}{D}) \int_{\mathcal{I}} f_{semi}(x)dx| < \delta(n + \frac{n}{D}) \frac{k}{\sqrt{k-1}} |\mathcal{I}| + 2t, \qquad (1)$$

again with probability at least $1 - o(1/n)$.

Invariant Subspace Perturbations

- Eigenvectors under perturbation require more careful treatment
- Dependent on separation of spectrum

Example

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \implies \sigma(A) = \{1\}$, $V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Let $\widetilde{A} = \begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix} \implies \sigma(\widetilde{A}) = \{1-\epsilon, 1+\epsilon\}$, $\widetilde{V} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

- Introduced by Davis 1963 for single eigenvector
- Generalized by Davis and Kahan 1970
- “Matrix Perturbation Theory” by Stewart and Sun 1990
- Localization of QR, LU, and Cholesky by Krishtal, Strohmer, and Wertz
- Studied on graphs by Rajapakse 2013

Invariant Subspace Perturbations (cont.)

Theorem (Davis, 1963)

Let $A, E \in \mathbb{C}^{n\times n}$. Let (λ, x) be an eigenpair of A such that

$$\mathrm{sep}(\lambda, \sigma(A) \setminus \lambda) = \min\{|\lambda - \gamma| : \gamma \in \sigma(A) \setminus \lambda\} = \delta.$$

Let

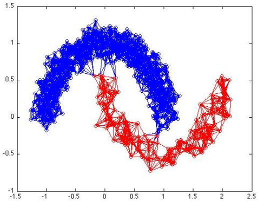
- *P be a spectral projector of A such that $Px = x$*
- *P' be the corresponding spectral projector of $A + E$, and*
- *$\overline{P'}z = z - P'z$.*

Then if $\|E\| \leq \epsilon \leq \delta/2$,

$$\|\overline{P'}P\| \leq \frac{\epsilon}{\delta - \epsilon}.$$

Poor Prediction Using Existing Theory

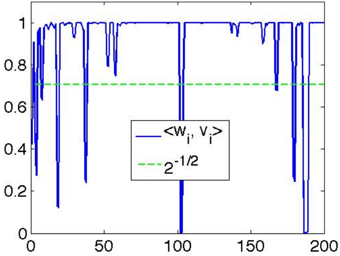
- Consider two moons example
  - $\{(\widetilde{\lambda}_i, v_i)\}$ eigenpairs of weakly connected graph L_G
  - $\{(\lambda_i, w_i)\}$ eigenpairs of disjoint graph L_H
  - Generate L_H by removing off block diagonal entries



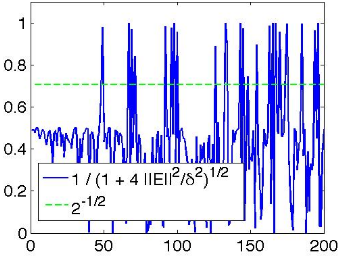
- $\sigma(L_H) \in [0, 2]$ is not sufficiently separated for existing theory
- Assumptions in literature are too strict for problem
- Also we are interested in localization, not angle

Poor Prediction Using Existing Theory (cont.)

Actual Vector Angles



Predicted Vector Angles



Eigengap Dependence on Similar Eigenvectors

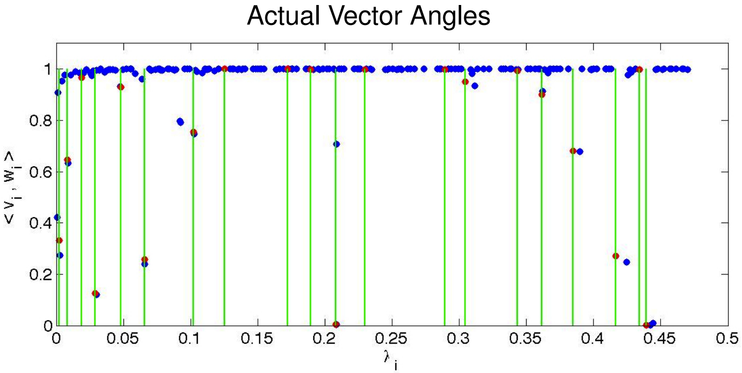
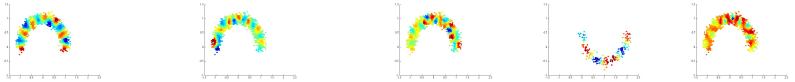


Figure: Green Line Denotes Eigenvector of L_H Concentrated on Smaller C_2 Cluster

Eigengap Dependence on Similar Eigenvectors (cont.)

- Problem eigenvector w_i follows pattern
  - $\text{supp}(w_i) \subset C_2$
  - $\text{supp}(w_{i-1}) \subset C_1$
  - $\text{supp}(w_{i+1}) \subset C_1$
- This is case for most eigenvectors from smaller C_2 cluster
  - $|C_1| = D|C_2| \implies \frac{D}{D+1}$ eigenvectors of L_H concentrated on C_1
- Consider eigenvector w_{25} of L_H as example

$\lambda_{20} = 0.0287$ ... $\lambda_{25} = 0.0371$ ... $\lambda_{30} = 0.0479$ $\lambda_{31} = 0.0481$ $\lambda_{32} = 0.0494$



Eigenvector Localization

Theorem (C., 2014)

*Let $L_H \in \mathbb{R}^{n\times n}$ be symmetric with eigendecomposition $L_H = V\Sigma V^*$.
Let (λ_i, v_i) be an eigenpair of L_H.
Partition $V = [V_1, V_2, v_i, V_3, V_4]$ where $V_2, V_3 \in \mathbb{R}^{n\times s}$.
Moreover, assume $\exists C \subsetneq \{1, ..., n\}$ such that $supp(v_i) \subset C$ and $supp(v_j) \subset C$ where v_j is a column of V_2, V_3.
Let $(\widetilde{\lambda}, x)$ an eigenvector of the perturbed matrix $L_G = L_H + E$, where $x = [x_1, ..., x_n]$. Then*

$$\sum_{j\in C^c} |x_j|^2 \leq \frac{\|(\widetilde{\lambda} - \lambda_i)x - Ex\|_2^2}{\min(\lambda_i - \lambda_{i-s}, \lambda_{i+s} - \lambda_i)^2}.$$

- Apply SVD Theorem to symmetric matrix $L_H - \lambda_i I$
  - SVD equivalence with eigendecomposition up to parity

Singular Vector Localization

Theorem (C., 2014)

Let $A \in \mathbb{R}^{n \times n}$ with SVD $A = U\Sigma V^*$. Partition

$$V = [V_1, V_2, v_n],$$

where $v_n \in \mathbb{R}^n$, $V_2 \in \mathbb{R}^{n \times s}$.
Moreover, assume $\exists C \subsetneq \{1, ..., n\}$ such that

$$\mathrm{supp}(v_i) \subset C \quad \text{for} \quad i \in \{n-s, ..., n\}.$$

Let $x \in \mathbb{R}^n$ such that $\|x\|_2 = 1$. Then

$$\sum_{i \in C^c} |x_i|^2 \leq \frac{\|Ax\|_2^2 - \|Av_n\|_2^2}{\sigma_{n-s-1}^2(A) - \sigma_n^2(A)}.$$

Sketch of Proof for SVD Localization

Assume $x = V_1c_1 + V_2c_2 + v_nc_3$.
Bound

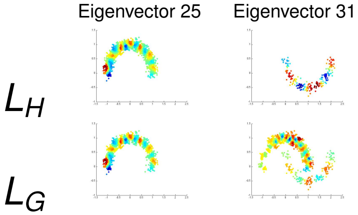
$$\|Ax\|_2^2 = \|U_1\Sigma_1V_1^*x + U_2\Sigma_1V_2^*x + u_n\sigma_nv_n^*x\|_2^2$$
$$\implies \|Ax\|_2^2 - \|Av_n\|_2^2 \geq (\sigma_{n-s-1}^2 - \sigma_n^2)\|c_1\|_2^2$$
$$\implies \|c_1\|_2^2 \leq \frac{\|Ax\|_2^2 - \|Av_n\|_2^2}{\sigma_{n-s-1}^2 - \sigma_n^2}.$$

Using the localization of V_2,

$$\sum_{i\in C^c}|x_i|^2 \leq \sum_{i=1}^{n}\sum_{j=1}^{n-s-1}|(V_1)_{i,j}c_j|^2$$
$$= \sum_{j=1}^{n-s-1}|c_j|^2$$
$$\leq \frac{\|Ax\|_2^2 - \|Av_n\|_2^2}{\sigma_{n-s-1}^2 - \sigma_n^2}.$$

Norbert Wiener Center for Harmonic Analysis and Applications

Eigenvector Localization Conclusions

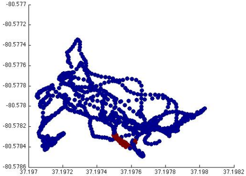


- Important notes from Theorem
  - Theorem implies 124 of 180 eigenvectors supported on C_1 remain concentrated
  - Only 3 of 20 eigenvectors supported on C_2 remain concentrated
  - Makes determining inter-cluster differences difficult for small clusters

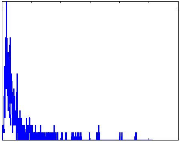
Outline

Nuclear Data

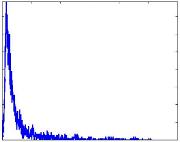
- Wish to detect anomalous material that emit radiation
- Build LE graph from radiological spectra



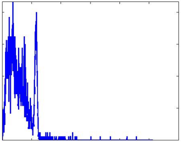
Background 1



Background 2



Anomalous Material

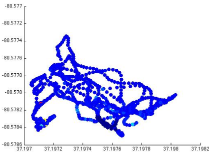


Data courtesy of Kevin Kochersberger, Virginia Tech

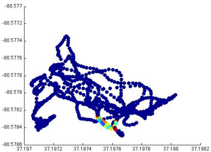
Anomalous Clusters Washed Out

- Use simple L^2 distance to create binary similarity kernel, 10 nearest neighbors
- $\#$ background measurements $= 1137$, $\#$ anomalous measurements $= 23$
  - $D = 49.4 \implies$ energy on anomalous cluster shows up $1:50$ times at best
  - All other eigenvectors (not shown) are only noise

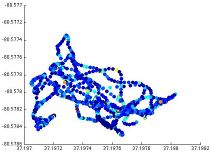
Eigenvector 1



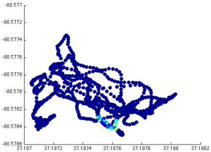
Eigenvector 3



Random Eigenvector



Eigenvector 81



Conclusions

- Takeaways:
  - Small clusters in disjoint graphs appear rarely in LE feature vectors
    - Proportional to number of data points in cluster
  - LE eigenvectors concentrated on small cluster rarely remain localized as graph becomes connected
  - Eigenvectors concentrated on larger cluster almost always remain localized
    - Leads to points on small cluster being forced to zero
  - Phenomenon is supported by simulated and real-world data
- Future Directions:
  - Upper bound on localization theory that doesn't require $A + E$ eigenvector
  - Theory for anomalous clusters that are smaller than k data points
  - Alter selection to subset of "low-frequency" eigenfunctions
    - Subset of indices originally introduced by Jones, Maggioni, Schul 2010

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Thank you!

Extra Slides

Eigenvalues of Regular Graph

Theorem (McKay, 1981)

Let $X_1, X_2, \ldots$ be a sequence of random k-regular graphs with adjacency matrices $A_1, A_2, \ldots$.
Let the family $\{X_i\}$ satisfy

1. $n(X_i) \to \infty$
2. $c_k(X_i)/n(X_i) \to 0.$

Then the empirical spectral distribution

$$F_n(x) = |\{i : \lambda_i \left(\frac{1}{\sqrt{k-1}} A_n \right) < x\}|/n$$

converges pointwise to the semicircle law

$$f_{semi}(x) = \frac{1}{2\pi}\sqrt{4 - x^2}, \quad -2 < x < 2.$$

Invariant Subspace Perturbations (cont.)

Theorem (Davis, 1963)

Let $A, E \in \mathbb{C}^{n\times n}$*. Let* (λ, x) *be an eigenpair of* A *such that*

$$\mathrm{sep}(\lambda, \sigma(A) \setminus \lambda) = \min\{|\lambda - \gamma| : \gamma \in \sigma(A) \setminus \lambda\} = \delta.$$

Let

- P *be a spectral projector of* A *such that* $Px = x$
- P' *be the corresponding spectral projector of* $A + E$*, and*
- $\overline{P'}z = z - P'z$.

Then if $\|E\| \leq \epsilon \leq \delta/2$,

$$\|\overline{P'}P\| \leq \frac{\epsilon}{\delta - \epsilon}.$$

Invariant Subspace Perturbations (cont.)

Theorem (Stewart, 1973)

Let $A, E \in \mathbb{C}^{n\times n}$. *Let* $X = [X_1, X_2]$ *be a unitary matrix with* $X_1 \in \mathbb{C}^{n\times l}$, *and suppose* $\mathcal{R}(X_1)$ *is an invariant subspace of* A. *Let*

$$X^*AX = \begin{pmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{pmatrix}, \quad X^*EX = \begin{pmatrix} E_{1,1} & E_{1,2} \\ E_{2,1} & E_{2,2} \end{pmatrix}.$$

Let $\delta = \operatorname{sep}(A_{1,1}, A_{2,2}) - \|E_{1,1}\| - \|E_{2,2}\|$. *Then if*

$$\frac{\|E_{2,1}\|(\|A_{1,2}\| + \|E_{1,2}\|)}{\delta^2} \leq \frac{1}{4},$$

there is a matrix P *satisfying* $\|P\| \leq 2\frac{\|E_{2,1}\|}{\delta}$ *such that*

$$\widetilde{X_1} = (X_1 + X_2P)(I + P^*P)^{-1//2}$$

is an invariant subspace of $A + E$.