Invariant Subspace Perturbations or: How I Learned to Stop Worrying and Love Eigenvectors

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Introduction to Eigenvector Perturbation

Eigenvalue Separation and Rigorous Arguments

Eigenvalue Concentration
Outline

1. Introduction to Eigenvector Perturbation
2. Eigenvalue Separation and Rigorous Arguments
3. Eigenvalue Concentration
Eigenvector decomposition is ubiquitous in mathematics
- Principle Component Analysis
- Quantum States
- Fourier Analysis
- Spectral Graph Theory

Most Common Problem in Mathematics
Find $$(\lambda, \nu)$$ such that

$$A\nu = \lambda\nu$$

One imagines computers made this problem trivial
- $$[U, S] = \text{eig}(A)$$

Question: What happens to eigenpairs if matrix has tiny errors

$$\tilde{A} = A + E$$
Answer for eigenvalues: You’re fine and they behave rather continuously
  - Weyl’s Inequality

Answer for eigenvectors:

Problem!
Eigenvector Perturbations

- Eigenvectors under perturbation require careful treatment
- Dependent on separation of spectrum

**Example**

Let $A = \begin{pmatrix} 1 & -\epsilon \\ 0 & 1 + \epsilon \end{pmatrix} \implies \sigma(A) = \{1 - \epsilon, 1 + \epsilon\}$, $V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Let $\tilde{A} = \begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix} \implies \sigma(\tilde{A}) = \{1 - \epsilon, 1 + \epsilon\}$, $\tilde{V} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

- $V$ and $\tilde{V}$ are as far apart as possible
Eigenvector Perturbations Geometrically

Problem is that image of $A$ is rotationally symmetric
Stable Eigenvector Geometrically

Lack of symmetry allows for robust perturbations
Eigenvector Perturbations

- Separation of spectrum creates stable perturbations

**Example**

Let \( B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \) \( \Rightarrow \) \( \sigma(B) = \{1, 2\} \), \( V = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \).

Let \( \tilde{B} = \begin{pmatrix} 1 & \epsilon \\ \epsilon & 2 \end{pmatrix} \) \( \Rightarrow \) \( \sigma(\tilde{B}) = \left\{ \frac{3 - \sqrt{1 + 16\epsilon^2}}{2}, \frac{3 + \sqrt{1 + 16\epsilon^2}}{2} \right\} \),

\[
\tilde{V} = \begin{pmatrix} 0.995 & 0.099 \\ -0.099 & 0.995 \end{pmatrix}
\] for \( \epsilon = .1 \)

- Rotation between \( V \) and \( \tilde{V} \) is \( \approx 5^\circ \)
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Consider symmetric matrices $A, E \in \mathbb{R}^{n \times n}$.

Eigenvectors of $A + E$ depend on separation of spectrum $\sigma(A)$.

**Definition (Separation of Spectrum)**

The separation of the spectrum $\sigma(A)$ at $\lambda$ is

$$\text{sep}(\lambda, \sigma(A) \setminus \lambda) = \min\{|\lambda - \gamma| : \gamma \in \sigma(A) \setminus \lambda\}$$
Theorem (Davis, 1963)

Let $A, E \in \mathbb{C}^{n \times n}$ be Hermetian. Let $(\lambda, x)$ be an eigenpair of $A$ such that

$$\text{sep}(\lambda, \sigma(A) \setminus \lambda) = \delta.$$ 

Let

- $P$ be a spectral projector of $A$ such that $Px = x$
- $P'$ be the corresponding spectral projector of $A + E$, and
- $\overline{P'}$ be the orthogonal complement $\overline{P'} z = z - P' z$.

Then if $\|E\| \leq \epsilon \leq \delta/2$,

$$\|\overline{P'} P\| \leq \frac{\epsilon}{\delta - \epsilon}.$$
Side Note on Advisers:
Similar Perturbation Theorems

Davis, Kahan (1970): Clustered Subspaces are Preserved

\[ \sigma(A) \setminus \Lambda \]
\[ \Lambda = \{ \lambda_1, \ldots, \lambda_s \} \]
\[ \sigma(A) \setminus \Lambda \]

\[ \delta \]
\[ \varepsilon \]
\[ \varepsilon \]

Stewart (1973): Take Direction of Error Into Account
Concentration As Opposed to Angle Similarity

- Previous theory defined similarity by angle $\Theta[V, \tilde{V}]$
  - Not only way to consider “similarity”
- Can also consider eigenvector “localization”
- Important when:
  - $\sigma(A)$ has high density in interval
  - $A$ is adjacency matrix for network graph

Plot of Eigenvectors for $A \in \mathbb{R}^{1000 \times 1000}$

$\lambda_{20} = 0.0287$  ...  $\lambda_{25} = 0.0371$  ...  $\lambda_{30} = 0.0479$  $\lambda_{31} = 0.0481$  $\lambda_{32} = 0.0494$
Concentration Perturbation

Plot of Eigenvectors for $A, \tilde{A} \in \mathbb{R}^{1000 \times 1000}$, $\tilde{A} = A + E$
**Theorem (C., 2014)**

Let $A \in \mathbb{R}^{n \times n}$ be symmetric with eigendecomposition $A = V\Sigma V^*$. Let $(\lambda_i, v_i)$ be an eigenpair of $A$. Assume

- Partition $V = [V_1, V_2, v_i, V_3, V_4]$ where $V_2, V_3 \in \mathbb{R}^{n \times s}$, ordered by $\lambda_1 \leq ... \leq \lambda_n$

- $\exists C \subsetneq \{1, ..., n\}$ such that $\text{supp}(v_i) \subset C$ and $\text{supp}(v_j) \subset C$ where $v_j$ is a column of $V_2, V_3$.

- Let $(\tilde{\lambda}, x)$ an eigenvector of the perturbed matrix $\tilde{A} = A + E$, where $x = [x_1, ..., x_n]$.

Then

$$\sum_{j \in C^c} |x_j|^2 \leq \frac{\|(\tilde{\lambda} - \lambda_i)x - Ex\|^2}{\min(\lambda_i - \lambda_{i-s}, \lambda_{i+s} - \lambda_i)^2}.$$
Conclusions

- Eigenvector perturbation depends on inverse of spectrum separation
- Led to Nobel Prize in Physics for particle localization (Anderson 1977)
- Concentration / localization is lesser restriction on eigenvectors
  - Depends on cluster of vectors concentrated in similar area
- Applications in eigenstate localization on data-dependent graphs