On the Spectral Synthesis Problem

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- Suppose $\mathcal{G}$ is a locally compact abelian group. Let $\Gamma := \widehat{\mathcal{G}}$, the *dual group of* $\mathcal{G}$. We call $\gamma \in \Gamma$ a *character*.
- For $f \in L^1(\mathcal{G})$, let $\hat{f} : \Gamma \to \mathbb{C}$ denote the *Fourier transform of* f:

$$\hat{f}(\gamma) := \int_{\mathcal{G}} f(x)\gamma(x)d\mu(x),$$

where μ is the Haar measure on $\mathcal{G}$.
- Let

$$A(\Gamma) := \left\{ \hat{f} \mid f \in L^1(\mathcal{G}) \right\}$$

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- The space of *pseudomeasures* on Γ, denoted $A'(\Gamma)$, is the Banach dual of $A(\Gamma)$. Alternatively, we may set $A'(\Gamma) := \left\{ \hat{F} \mid F \in L^\infty(\mathcal{G}) \right\}$.
- The duality between $A(\Gamma)$ and $A'(\Gamma)$ is as follows: for all $T \in A'(\Gamma)$ and all $\phi \in A(\Gamma)$, we define

$$\langle T, \phi \rangle = \langle \hat{T}, \hat{\phi} \rangle.$$

- A closed set $\Lambda \subset \Gamma$ is said to be a set of *spectral synthesis*, *S-set* for short, if: $\forall\ T \in A'(\Gamma)$, $\forall \phi \in A(\Gamma)$:

$$\left.\begin{array}{r} \mathrm{supp}(T) \subset \Lambda \\ \phi = 0 \text{ on } \Lambda \end{array}\right\} \qquad \Longrightarrow \langle T, \phi \rangle = 0.$$

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- The following notion appears in the works of Beurling on balayage, dating as early as the 1940's.

Definition

We say a closed subset $\Lambda \subset \Gamma$ is an *B-Set* if: $\forall\, \mu \in M(\mathcal{G})$, $\forall f \in \mathcal{C}_b(\mathcal{G})$:

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Theorem

(JJB, JMM) For every $\Lambda \subset \Gamma$, Λ is an S-set iff it is a B-set.

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- The study of spectral synthesis is closely related to both the ideal structure of $A(\Gamma)$, and the theory of Fourier series for L^∞ functions.
- For $\Lambda \subset \Gamma$ closed, we set

$$j(\Lambda) := \{\phi \in A(\Gamma) \mid \mathrm{supp}(\phi) \cap \Lambda = \emptyset\}.$$
$$k(\Lambda) := \{\phi \in A(\Gamma) \mid \phi|_\Lambda \equiv 0\}.$$

- $\overline{j(\Lambda)}$ is the smallest closed ideal containing all the elements of $A(\Gamma)$ vanishing on Λ, and $k(\Lambda)$ is the largest such closed ideal.

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- How to formulate a notion of Fourier series for $F \in L^\infty(\mathcal{G})$?

Definition

Let $\mathcal{G}$ be a locally compact abelian group with dual group Γ. For a translation-invariant, weak-$*$ closed subspace $\mathcal{J} \subset L^\infty(\mathcal{G})$, we define the *spectrum of $\mathcal{J}$* to be:

$$\mathrm{sp}(\mathcal{J}) := \{\gamma \in \Gamma \mid (\gamma, \cdot) \in \mathcal{J}\}.$$

Let $\mathcal{J}_{\mathrm{sp}}$ be the weak-$*$ closure of the span of $\mathrm{sp}\mathcal{J}$. We say the elements of $\mathcal{J}$ are *synthesizable* if $\mathcal{J} = \mathcal{J}_{\mathrm{sp}}$.
In particular, $F \in L^\infty(\mathcal{G})$ is *synthesizable* if $F \in \mathcal{J}^F_{sp}$, where $\mathcal{J}^F$ is generated by the translates of F.

- In short, $F \in L^\infty(\mathcal{G})$ is synthesizable if it can be written as a weak-$*$ convergent linear combination of characters in the spectrum of F. This generalizes the notion of Fourier series.

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Theorem

TFAE for $\Lambda \subset \Gamma$ closed:

1. *Λ is an S-set.*
2. $\overline{j(\Lambda)} = k(\Lambda)$.
3. *$\Lambda = sp(\mathcal{J})$ for a unique translation-invariant subspace $\mathcal{J} \subset L^\infty(\mathcal{G})$.*

- The equivalence (1) $\Leftrightarrow$ (2) relates the spectral synthesis problem to the ideal structure of $A(\Gamma)$.
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- Examples of S-sets include:
    1. $\emptyset$; this is a direct consequence of Wiener's Lemma for the inversion of Fourier series.
    2. The Cantor set in $\mathbb{R}$ (Herz).
    3. $S^1 \subset \mathbb{R}^2$ (Herz).
    4. $\{\gamma\} \subset \Gamma$, for any Γ.
    5. All closed $\Lambda \subset \Gamma$ for Γ discrete.
    6. Star-shaped sets in $\mathbb{R}^d$.
    7. $\Lambda \subset \Gamma$ such that $\Lambda + \Lambda \subset \Lambda$ and such that $0 \in \overline{\Lambda^\circ}$.

- In 1948, L. Schwartz proved that $S^2 \subset \mathbb{R}^3$, the unit sphere in three dimensions, is *not* a set of spectral synthesis.

- It was later shown by Malliavin that every non-discrete Γ contains a closed set which is not of spectral synthesis.

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- **Open Question**: Is the union of two sets of spectral synthesis itself of spectral synthesis?

- For certain *related but weaker* notions of spectral synthesis, this has been shown.

- There is active work on spectral synthesis in the context of algebraic groups (Szèkelyhidi), operator algebras (Ludwig, Turowska et al.) and manifolds in $\mathbb{R}^d$. We shall focus on this last subject for the remainder of the talk.

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- Schwartz's example of a set that is not of spectral synthesis is a manifold in $\mathbb{R}^3$.
- What role does differential geometry play in whether or not a manifold will be of spectral synthesis? A big one, as it turns out.
- In the context of studying the spectral synthesis properties of manifolds in Euclidean space, it is interesting to study a slightly weaker notion of spectral synthesis, which we shall call *weak spectral synthesis*.

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Definition

A closed set $\Lambda \subset \mathbb{R}^d$ is a set of *weak spectral synthesis* if for every $f \in A(\mathbb{R}^d)$ vanishing on Λ, there exists a sequence $\{\phi_n\}_{n=1}^{\infty} \subset \mathcal{S}(\mathbb{R}^d)$ of Schwartz functions vanishing on Λ that converges to f in the $A(\mathbb{R}^d)$ norm.

- Why call this "spectral synthesis?"

Theorem

(JJB, JMM)
$\Lambda \subset \mathbb{R}$ is an S-set if and only if for every $f \in A(\mathbb{R})$ vanishing on Λ, there exists a sequence $\{\phi_n\}_{n=1}^{\infty} \subset \mathcal{S}(\mathbb{R})$ of Schwartz functions vanishing on a neighborhood of Λ that converges to f in the $A(\mathbb{R})$ norm.

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- With this slightly weaker formulation of spectral synthesis, things get very interesting. It has been shown by Varopoulos that $S^{d-1} \subset \mathbb{R}^d$ is a set of weak spectral synthesis for all $d \geq 1$.

- There are generalizations of Malliavin's theorem, showing the existence of subsets of $\mathbb{R}^d$ that are not of weak spectral synthesis, for all $d \geq 1$. These sets lack obvious structure and regularity, and are not manifolds.

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- With this slightly weaker formulation of spectral synthesis, things get very interesting. It has been shown by Varopoulos that $S^{d-1} \subset \mathbb{R}^d$ is a set of weak spectral synthesis for all $d \geq 1$.
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- No. Domar made substantial contributions to understanding what conditions on a manifold ensure it is a set of weak spectral synthesis.
- Let us first examine the case of curves, which is some sense the simplest class of manifolds. The notion of *curvature* plays a crucial role here:

Definition

For a C^2 plane curve $\gamma = (\gamma_1, \gamma_2) : [a, b] \to \mathbb{R}^2$, the *planar curvature* at $s \in [a, b]$ is given by the formula

$$\kappa(s) := \frac{\gamma_1'(s)\gamma_2''(s) - \gamma_1''(s)\gamma_2'(s)}{((\gamma_1'(s))^2 + (\gamma_2'(s))^2)^{\frac{3}{2}}}.$$

Theorem

(Domar) Suppose $\Lambda \subset \mathbb{R}^2$ is the graph of a simple C^2 curve that is of non-vanishing planar curvature everywhere. Then Λ is a set of weak spectral synthesis.

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- Using a substantially modified technique, Domar also proved the following:

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(Domar) There exists a $\mathcal{C}^\infty$ planar curve whose graph $\Lambda \subset \mathbb{R}^2$ is not of spectral synthesis.

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Definition

A closed set $\Lambda \subset \mathbb{R}^d$ is said to have the *restricted cone property at a point* $x_0 \in \mathbb{R}^d$ if there exists a neighborhood U_0 of x_0 and a cone K defined by:

$$K := \left\{ x \in \mathbb{R}^d \,\middle|\, (1-\delta)\|x\| \le \langle x, z \rangle \le \delta \right\},$$

where $0 < \delta < 1$, $z \in S^{d-1}$, such that:

$$x - K \subset \Lambda,$$

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- Intuitively, a closed set Λ has the restricted cone property if for every point in $x_0 \in \Lambda$, we can put a cone that is either long and thin or short and wide, with vertex at x_0, entirely into the set Λ.

Theorem

(Domar) Assume $d \geq 2$ and that M is a $(d-1)$-dimensional C^∞ manifold without multiple points and with non-vanishing Gaussian curvature. Suppose $\Lambda \subset M$ is compact. If Λ has the restricted cone property, then it is a set of weak spectral synthesis.

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Thank you for your time!