Scalable frames

Kasso Okoudjou
joint with X. Chen, G. Kutyniok, F. Philipp, R. Wang

Department of Mathematics
&
Norbert Wiener Center
University of Maryland, College Park

The Math/Stat Department Colloquium
American University, Washington, DC
Tuesday November 18, 2014

Outline

1. Finite frames theory
    - Motivations and definition
    - Tight frames
    - Frame potential
2. Scalable frames
    - Transforming a frame into a tight frame
    - Some generic properties of scalable frames
    - Characterization of scalable frames
    - Fritz John's ellipsoid theorem and scalable frames
3. References

A standard problem

Question

Let $\Phi = \{\varphi_i\}_{i=1}^M \subset \mathbb{R}^N$ be a complete set. Recover x from $\hat{y}$ given by $\hat{y}$ given by

$$\hat{y} = \Phi^* x + \eta,$$

where Φ is the $N \times M$ matrix whose k^{th} column is φ_k, and η is an error (noise).

Solution

Need to design "good" measurement matrix Φ, e.g., Φ should lead to reconstruction methods that are robust to erasures and noise.

A standard problem

Question

Let $\Phi = \{\varphi_i\}_{i=1}^M \subset \mathbb{R}^N$ *be a complete set. Recover* x *from* $\hat{y}$ *given by* $\hat{y}$ *given by*

$$\hat{y} = \Phi^* x + \eta,$$

where Φ *is the* $N \times M$ *matrix whose* k^{th} *column is* φ_k*, and* η *is an error (noise).*

Solution

Need to design "good" measurement matrix Φ*, e.g.,* Φ *should lead to reconstruction methods that are robust to erasures and noise.*

Minimal requirements on the measurement matrix

Fact

$\Phi = \{\varphi_i\}_{i=1}^M \subset \mathbb{K}^N$ *is complete* $\iff \exists A > 0:$

$$A\|x\|^2 \leq \sum_{i=1}^M |\langle x, \varphi_i\rangle|^2 \quad \textit{for all } x \in \mathbb{K}^N$$

Clearly, there exists $B > 0$, *e.g.,* $B = \sum_{i=1}^M \|\varphi_i\|^2$ *such that*

$$\sum_{i=1}^M |\langle x, \varphi_i\rangle|^2 \leq B\|x\|^2 \quad \textit{for all } x \in \mathbb{K}^N.$$

Definition of finite frames

Definition

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. $\{\varphi_i\}_{i=1}^M \subset \mathbb{K}^N$ is called a *finite frame for* $\mathbb{K}^N$ if $\exists\, 0 < A \leq B$:

$$A\|x\|^2 \leq \sum_{i=1}^{M} |\langle x, \varphi_i\rangle|^2 \leq B\|x\|^2, \quad \text{for all } x \in \mathbb{K}^N. \tag{1}$$

If $A = B$, then $\{\varphi_i\}_{i=1}^M \subset \mathbb{K}^N$ is called *a finite tight frame for* $\mathbb{K}^N$.

Frame operator & Reconstruction formulas

- For $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{K}^N$ let $\Phi = \begin{bmatrix} \varphi_1 & \varphi_2 & \dots & \varphi_M \end{bmatrix}$.
- Φ is a frame $\iff$ $S = \Phi\Phi^*$ is positive definite.

$$x = S(S^{-1}x) = \sum_{i=1}^{M} \langle x, S^{-1}\varphi_i\rangle \varphi_i = \sum_{i=1}^{M} \langle x, \varphi_i\rangle S^{-1}\varphi_i$$

- $\widetilde{\Phi} = \{\tilde{\varphi}_i\}_{i=1}^M = \{S^{-1}\varphi_i\}_{i=1}^M$ is the *canonical dual frame.*
- $A_{opt} = \lambda_{min}(S)$ and $B_{opt} = \lambda_{max}(S)$. The condition number of the frame is

$$\kappa(\Phi) = \lambda_{max}(S)/\lambda_{min}(S) \geq 1.$$

Frame operator & Reconstruction formulas

- For $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{K}^N$ let $\Phi = \begin{bmatrix} \varphi_1 & \varphi_2 & \dots & \varphi_M \end{bmatrix}$.
- Φ is a frame $\iff$ $S = \Phi\Phi^*$ is positive definite.

$$x = S(S^{-1}x) = \sum_{i=1}^{M} \langle x, S^{-1}\varphi_i\rangle \varphi_i = \sum_{i=1}^{M} \langle x, \varphi_i\rangle S^{-1}\varphi_i$$

- $\widetilde{\Phi} = \{\tilde{\varphi}_i\}_{i=1}^M = \{S^{-1}\varphi_i\}_{i=1}^M$ is the *canonical dual frame*.
- $A_{opt} = \lambda_{min}(S)$ and $B_{opt} = \lambda_{max}(S)$. The condition number of the frame is

$$\kappa(\Phi) = \lambda_{max}(S)/\lambda_{min}(S) \geq 1.$$

Frame operator & Reconstruction formulas

- For $\Phi = \{\varphi_k\}_{k=1}^{M} \subset \mathbb{K}^N$ let $\Phi = \begin{bmatrix} \varphi_1 & \varphi_2 & \dots & \varphi_M \end{bmatrix}$.
- Φ is a frame $\iff$ $S = \Phi\Phi^*$ is positive definite.

$$x = S(S^{-1}x) = \sum_{i=1}^{M} \langle x, S^{-1}\varphi_i\rangle \varphi_i = \sum_{i=1}^{M} \langle x, \varphi_i\rangle S^{-1}\varphi_i$$

- $\widetilde{\Phi} = \{\tilde{\varphi}_i\}_{i=1}^{M} = \{S^{-1}\varphi_i\}_{i=1}^{M}$ is the *canonical dual frame*.
- $A_{opt} = \lambda_{min}(S)$ and $B_{opt} = \lambda_{max}(S)$. The condition number of the frame is

$$\kappa(\Phi) = \lambda_{max}(S)/\lambda_{min}(S) \geq 1.$$

Frame operator & Reconstruction formulas

- For $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{K}^N$ let $\Phi = \begin{bmatrix} \varphi_1 & \varphi_2 & \dots & \varphi_M \end{bmatrix}$.
- Φ is a frame $\iff$ $S = \Phi\Phi^*$ is positive definite.

$$x = S(S^{-1}x) = \sum_{i=1}^{M} \langle x, S^{-1}\varphi_i\rangle \varphi_i = \sum_{i=1}^{M} \langle x, \varphi_i\rangle S^{-1}\varphi_i$$

- $\widetilde{\Phi} = \{\tilde{\varphi}_i\}_{i=1}^M = \{S^{-1}\varphi_i\}_{i=1}^M$ is the *canonical dual frame*.
- $A_{opt} = \lambda_{min}(S)$ and $B_{opt} = \lambda_{max}(S)$. The condition number of the frame is

$$\kappa(\Phi) = \lambda_{max}(S)/\lambda_{min}(S) \geq 1.$$

Frame operator & Reconstruction formulas

- For $\Phi = \{\varphi_k\}_{k=1}^{M} \subset \mathbb{K}^N$ let $\Phi = \begin{bmatrix} \varphi_1 & \varphi_2 & \dots & \varphi_M \end{bmatrix}$.
- Φ is a frame $\iff$ $S = \Phi\Phi^*$ is positive definite.

$$x = S(S^{-1}x) = \sum_{i=1}^{M} \langle x, S^{-1}\varphi_i\rangle \varphi_i = \sum_{i=1}^{M} \langle x, \varphi_i\rangle S^{-1}\varphi_i$$

- $\widetilde{\Phi} = \{\tilde{\varphi}_i\}_{i=1}^{M} = \{S^{-1}\varphi_i\}_{i=1}^{M}$ is the *canonical dual frame*.
- $A_{opt} = \lambda_{min}(S)$ and $B_{opt} = \lambda_{max}(S)$. The condition number of the frame is

$$\kappa(\Phi) = \lambda_{max}(S)/\lambda_{min}(S) \geq 1.$$

The canonical dual frame

Lemma

Assume that $\Phi = \{\varphi_i\}_{i=1}^M \subset \mathbb{K}^N$ *is a frame, and that* $\{\tilde{\varphi}_i\}_{i=1}^M \subset \mathbb{K}^N$ *is the canonical dual frame. For each* $x \in \mathbb{K}^N$*,* $\sum_{i=1}^M |\langle x, \tilde{\varphi}_i\rangle|^2$ *minimizes* $\sum_{i=1}^M |c_i|^2$ *for all* $\{c_i\}_{i=1}^M$ *such that* $x = \sum_{i=1}^M c_i\varphi_i$.

Why frames?

Question

Let $\Phi = \{\varphi_i\}_{i=1}^M \subset \mathbb{R}^N$ be a unit norm frame, and assume we wish to recover x where we have access to $\hat{y}$ given by

$$\hat{y} = \Phi^* x + \eta.$$

Solution

If no assumption is made about η we can just minimize $\|\Phi^* x - \hat{y}\|_2$. This leads to

$$\hat{x} = (\Phi^\dagger)^* \hat{y} = \sum_{i=1}^M (\langle x, \varphi_i \rangle + \eta_i)\tilde{\varphi}_i.$$

Why frames?

Question

Let $\Phi = \{\varphi_i\}_{i=1}^M \subset \mathbb{R}^N$ *be a unit norm frame, and assume we wish to recover* x *where we have access to* $\hat{y}$ *given by*

$$\hat{y} = \Phi^* x + \eta.$$

Solution

If no assumption is made about η *we can just minimize* $\|\Phi^* x - \hat{y}\|_2$. *This leads to*

$$\hat{x} = (\Phi^\dagger)^* \hat{y} = \sum_{i=1}^{M} (\langle x, \varphi_i \rangle + \eta_i) \tilde{\varphi}_i.$$

Finite unit norm tight frames

Definition

A tight frame $\{\varphi_i\}_{i=1}^M \subset \mathbb{K}^N$ with $\|\varphi_k\| = 1$ for each k is called a *finite unit norm tight frame (FUNTF)* for $\mathbb{K}^N$. In this case, the frame bound is $A = M/N$.

Remark

Tight frames and FUNTFs can be considered optimally conditioned frames since the condition number of their frame operator is unity.

Finite unit norm tight frames

Definition

A tight frame $\{\varphi_i\}_{i=1}^M \subset \mathbb{K}^N$ with $\|\varphi_k\| = 1$ for each k is called a *finite unit norm tight frame (FUNTF)* for $\mathbb{K}^N$. In this case, the frame bound is $A = M/N$.

Remark

Tight frames and FUNTFs can be considered optimally conditioned frames since the condition number of their frame operator is unity.

Reconstruction formulas for tight frames

- If Φ is a tight frame then $S = AI$ and $x = \frac{1}{A}\sum_{k=1}^{M}\langle x, \varphi_k\rangle\varphi_k$.
- If $\Phi = \{\varphi_k\}_{k=1}^{M} \subset \mathbb{K}^N$ is a frame then $\{S^{-1/2}\varphi_k\}_{k=1}^{M}$ is a tight frame.

Example

- Any (properly normalized) N rows from the $M \times M$ DFT matrix is a tight frame.
- Every tight frame of M vectors in $\mathbb{K}^N$ is obtained from an orthogonal projection of an ONB in $\mathbb{K}^M$ onto $\mathbb{K}^N$.

Reconstruction formulas for tight frames

- If Φ is a tight frame then $S = AI$ and $x = \frac{1}{A}\sum_{k=1}^{M}\langle x, \varphi_k\rangle\varphi_k$.
- If $\Phi = \{\varphi_k\}_{k=1}^{M} \subset \mathbb{K}^N$ is a frame then $\{S^{-1/2}\varphi_k\}_{k=1}^{M}$ is a tight frame.

Example

- Any (properly normalized) N rows from the $M \times M$ DFT matrix is a tight frame.
- Every tight frame of M vectors in $\mathbb{K}^N$ is obtained from an orthogonal projection of an ONB in $\mathbb{K}^M$ onto $\mathbb{K}^N$.

Examples of frames

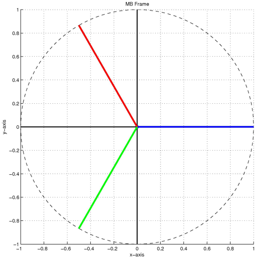

Figure : The MB Frame

Why tight frames?

Assume each component of η has zero mean and variance σ^2, and that η_i and η_j are uncorrelated for $i \neq j$. Then

$$x - \hat{x} = \sum_{i=1}^{M} \langle x, \varphi_i \rangle \tilde{\varphi}_i - \sum_{i=1}^{M} (\langle x, \varphi_i \rangle + \eta_i) \tilde{\varphi}_i = - \sum_{i=1}^{M} \eta_i \tilde{\varphi}_i.$$

Consequently,

$$MSE = \frac{1}{N} E \|x - \hat{x}\|^2 = \frac{1}{N} \operatorname{Trace}(S^{-1}) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\lambda_i}$$

where $\{\lambda_i\}_{i=1}^N$ is the spectrum of S.

Theorem (Goyal, Kovačević, and Kelner (2001))

The MSE is minimum if and only if the frame Φ is tight.

Frames in applications

Example

- Quantum computing: construction of POVMs
- Spherical t-designs
- Classification of hyper-spectral data
- Quantization
- Phase-less reconstruction
- Compressed sensing.

Question

How to construct tight frames and/or FUNTFs?

Frames in applications

Example

- Quantum computing: construction of POVMs
- Spherical t-designs
- Classification of hyper-spectral data
- Quantization
- Phase-less reconstruction
- Compressed sensing.

Question

How to construct tight frames and/or FUNTFs?

Existence and characterization of FUNTFs

Theorem (Benedetto, Fickus (2003))

Let $M \in \mathbb{N}$ and $\Phi = \{\varphi_k\}_{k=1}^{M} \subset S^{N-1}$. The frame potential satisfies

$$FP(\Phi) = \sum_{i=1}^{M} \sum_{j=1}^{M} |\langle \varphi_i, \varphi_j \rangle|^2 \geq \max(M, N)\frac{M}{N}.$$

In particular,

- *If $M \leq N$, then $\min FP(\Phi) = M$. The minimizers are the orthonormal systems for $\mathbb{K}^N$ with M elements.*
- *If $M \geq N$, then $\min FP(\Phi) = \frac{M^2}{N}$. The minimizers are the FUNTFs for $\mathbb{K}^N$ with M elements.*

Existence and characterization of FUNTFs

Theorem (Benedetto, Fickus (2003))

Let $M \in \mathbb{N}$ *and* $\Phi = \{\varphi_k\}_{k=1}^M \subset S^{N-1}$. *The frame potential satisfies*

$$FP(\Phi) = \sum_{i=1}^{M} \sum_{j=1}^{M} |\langle \varphi_i, \varphi_j \rangle|^2 \geq \max(M, N) \frac{M}{N}.$$

In particular,

- *If* $M \leq N$, *then* $\min FP(\Phi) = M$. *The minimizers are the orthonormal systems for* $\mathbb{K}^N$ *with* M *elements.*
- *If* $M \geq N$, *then* $\min FP(\Phi) = \frac{M^2}{N}$. *The minimizers are the FUNTFs for* $\mathbb{K}^N$ *with* M *elements.*

Proof

Proof.

$$FP(\{\varphi_k\}_{k=1}^M) = M + \sum_{k\neq\ell=1}^{M} |\langle\varphi_k, \varphi_\ell\rangle|^2 \geq M.$$

• So If $M \leq N$ the minimizers are exactly orthonormal systems and the minimum is M.

• Now assume $M \geq N$ and let $G = \Phi^*\Phi$. Then,

$$FP(\{\varphi_k\}_{k=1}^M) = Tr(G^2) = \sum_{k=1}^{N} \lambda_k^2$$

and, $trace(G) = \sum_{k=1}^{N} \lambda_k = M$. □

Proof

Proof.

$$FP(\{\varphi_k\}_{k=1}^M) = M + \sum_{k\neq\ell=1}^{M} |\langle \varphi_k, \varphi_\ell\rangle|^2 \geq M.$$

• So If $M \leq N$ the minimizers are exactly orthonormal systems and the minimum is M.
• Now assume $M \geq N$ and let $G = \Phi^*\Phi$. Then,

$$FP(\{\varphi_k\}_{k=1}^M) = Tr(G^2) = \sum_{k=1}^{N} \lambda_k^2$$

and, $trace(G) = \sum_{k=1}^N \lambda_k = M$. □

Proof (continued)

Proof.

Minimizing $FP(\{\varphi_k\}_{k=1}^M)$ is equivalent to minimizing

$$\sum_{k=1}^{N} \lambda_k^2 \quad such\, that \quad \sum_{k=1}^{N} \lambda_k = M.$$

Solution: $\lambda_k = M/N$ for all k.
Hence $S = \frac{M}{N} I_N$ where I_N is the identity matrix. The corresponding minimizers $\{\varphi_k\}_{k=1}^M$ are FUNTFs

$$x = \frac{N}{M} \sum_{k=1}^{M} \langle x, \varphi_k \rangle \varphi_k \quad \forall x \in \mathbb{K}^N.$$

Proof (continued)

Proof.

Minimizing $FP(\{\varphi_k\}_{k=1}^M)$ is equivalent to minimizing

$$\sum_{k=1}^N \lambda_k^2 \quad such\, that \quad \sum_{k=1}^N \lambda_k = M.$$

Solution: $\lambda_k = M/N$ for all k.

Hence $S = \frac{M}{N} I_N$ where I_N is the identity matrix. The corresponding minimizers $\{\varphi_k\}_{k=1}^M$ are FUNTFs

$$x = \frac{N}{M} \sum_{k=1}^M \langle x, \varphi_k \rangle \varphi_k \quad \forall x \in \mathbb{K}^N.$$

Proof (continued)

Proof.

Minimizing $FP(\{\varphi_k\}_{k=1}^M)$ is equivalent to minimizing

$$\sum_{k=1}^N \lambda_k^2 \quad such\,that \quad \sum_{k=1}^N \lambda_k = M.$$

Solution: $\lambda_k = M/N$ for all k.
Hence $S = \frac{M}{N} I_N$ where I_N is the identity matrix. The corresponding minimizers $\{\varphi_k\}_{k=1}^M$ are FUNTFs

$$x = \frac{N}{M} \sum_{k=1}^M \langle x, \varphi_k \rangle \varphi_k \quad \forall x \in \mathbb{K}^N.$$

Construction of FUNTFs

Fact

- *Numerical schemes such as gradient descent can be used to find minimizers of the frame potential and thus find FUNTFs.*
- *The spectral tetris method was proposed by Casazza, Fickus, Mixon, Wang, and Zhou (2011) to construct all FUNTFs. Further contributions by Krahmer, Kutyniok, Lemvig, (2012); Lemvig, Miller, Okoudjou (2012).*
- *Other methods (algebraic geometry) have been proposed by Cahill, Fickus, Mixon, Strawn.*

Construction of FUNTFs

Fact

- *Numerical schemes such as gradient descent can be used to find minimizers of the frame potential and thus find FUNTFs.*
- *The spectral tetris method was proposed by Casazza, Fickus, Mixon, Wang, and Zhou (2011) to construct all FUNTFs. Further contributions by Krahmer, Kutyniok, Lemvig, (2012); Lemvig, Miller, Okoudjou (2012).*
- *Other methods (algebraic geometry) have been proposed by Cahill, Fickus, Mixon, Strawn.*

Construction of FUNTFs

Fact

- *Numerical schemes such as gradient descent can be used to find minimizers of the frame potential and thus find FUNTFs.*
- *The spectral tetris method was proposed by Casazza, Fickus, Mixon, Wang, and Zhou (2011) to construct all FUNTFs. Further contributions by Krahmer, Kutyniok, Lemvig, (2012); Lemvig, Miller, Okoudjou (2012).*
- *Other methods (algebraic geometry) have been proposed by Cahill, Fickus, Mixon, Strawn.*

Main question

Question

Given a (non-tight) frame $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ can one transform Φ into a tight frame? If yes can this be done algorithmically and can the class of all frames that allow such transformations be described?

Solution

1. *A solution: The canonical tight frame $\{S^{-1/2}\varphi_k\}_{k=1}^M$. Involves the inverse frame operator.*
2. *What "transformations" are allowed?*

Choosing a transformation

Question

Given a (non-tight) frame $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ *can one find nonnegative numbers* $\{c_k\}_{k=1}^M \subset [0, \infty)$ *such that* $\widetilde{\Phi} = \{c_k \varphi_k\}_{k=1}^M$ *becomes a tight frame?*

Definition

Definition

Given $N \leq M$, a frame $\Phi = \{\varphi_k\}_{k=1}^{M}$ in $\mathbb{R}^N$ is *scalable* if there exists $\{x_k\}_{k=1}^{M}$ such that $\widetilde{\Phi_I} = \{x_k \varphi_k\}_{k=1}^{M}$ is a tight frame for $\mathbb{R}^N$.
More generally, given $N \leq m \leq M$, a frame $\Phi = \{\varphi_k\}_{k=1}^{M}$ in $\mathbb{R}^N$ is $m-$*scalable* if there exists a subset $\Phi_I = \{\varphi_k\}_{k\in I}$ with $\#I = m$, such that Φ_I is scalable.

Elementary properties

Lemma (G. Kutyniok, F. Philipp, E. K. Tuley, K. O. (2012))

1. *If $\Phi \subset \mathbb{R}^N$ is scalable frame if and only if $T(\Phi)$ is scalable for one (thus for all) orthogonal matrix T.*
2. *The set of scalable frames is closed in the set of all frames with M vectors.*

Fact

Let $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N \setminus \{0\}$ be a frame, with $M \geq N$, $\varphi_k \neq \varphi_\ell$ for $k \neq \ell$. Φ is scalable if and only if $\tilde{\Phi} = \{\pm\varphi_k/\|\varphi_k\|\}_{k=1}^M$ is scalable.

Elementary properties

Lemma (G. Kutyniok, F. Philipp, E. K. Tuley, K. O. (2012))

1. *If $\Phi \subset \mathbb{R}^N$ is scalable frame if and only if $T(\Phi)$ is scalable for one (thus for all) orthogonal matrix T.*
2. *The set of scalable frames is closed in the set of all frames with M vectors.*

Fact

Let $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N \setminus \{0\}$ be a frame, with $M \geq N$, $\varphi_k \neq \varphi_\ell$ for $k \neq \ell$. Φ is scalable if and only if $\tilde{\Phi} = \{\pm\varphi_k/\|\varphi_k\|\}_{k=1}^M$ is scalable.

Elementary properties

Lemma (G. Kutyniok, F. Philipp, E. K. Tuley, K. O. (2012))

1. *If $\Phi \subset \mathbb{R}^N$ is scalable frame if and only if $T(\Phi)$ is scalable for one (thus for all) orthogonal matrix T.*
2. *The set of scalable frames is closed in the set of all frames with M vectors.*

Fact

Let $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N \setminus \{0\}$ be a frame, with $M \geq N$, $\varphi_k \neq \varphi_\ell$ for $k \neq \ell$. Φ is scalable if and only if $\tilde{\Phi} = \{\pm\varphi_k/\|\varphi_k\|\}_{k=1}^M$ is scalable.

The scaling problem

$$\Phi = \{\varphi_i\}_{i=1}^M \text{ is } scalable \iff \exists \{c_i\}_{i=1}^M \subset [0,\infty) : \Phi C \Phi^T = I,$$

where $C = \text{diag}(c_i)$.

Scalable frames in $\mathbb{R}^2$

Question

Assume $M \geq 3$. When is

$$\varphi_k = \begin{pmatrix} \cos\theta_k \\ \sin\theta_k \end{pmatrix} \in S^1$$

with

$$0 = \theta_1 < \theta_2 < \theta_3 < \ldots < \theta_M < \pi$$

a scalable frame.

Scalable frames in $\mathbb{R}^2$

Solution

We need to solve

$$\Phi X^2 \Phi^T = \tilde{A} I_N$$

which is equivalent to finding a nontrivial nonnegative vector $Y = (y_k)_{k=1}^M \subset [0, \infty)$, *such that*

$$\Phi diag(Y) \Phi^T = I_N.$$

Scalable frames in $\mathbb{R}^2$

Solution

We must solve:

$$\begin{pmatrix} \sum_{k=1}^{M} y_k \cos^2\theta_k & \sum_{k=1}^{M} y_k \sin\theta_k \cos\theta_k \\ \sum_{k=1}^{M} y_k \sin\theta_k \cos\theta_k & \sum_{k=1}^{M} y_k \sin^2\theta_k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

or equivalently

$$\begin{cases} \sum_{k=1}^{M} y_k \sin^2\theta_k & = & 1 \\ \sum_{k=1}^{M} y_k \cos 2\theta_k & = & 0 \\ \sum_{k=1}^{M} y_k \sin 2\theta_k & = & 0. \end{cases}$$

Scalable frames in $\mathbb{R}^2$

Solution

For Φ to be scalable we must find a nonnegative vector $Y = (y_k)_{k=1}^{M}$ in the kernel of the matrix whose k^{th} column is

$$\begin{pmatrix} \cos 2\theta_k \\ \sin 2\theta_k \end{pmatrix}.$$

The first equation is just a normalization condition.

Scalable frames of 3 vectors in $\mathbb{R}^2$

Solution

We need to find non-trivial nonnegative vectors in the kernel of

$$\begin{pmatrix} 1 & \cos 2\theta_2 & \dots & \cos 2\theta_M \\ 0 & \sin 2\theta_2 & \dots & \sin 2\theta_M \end{pmatrix}. \tag{2}$$

Scalable frames of 3 vectors in $\mathbb{R}^2$

Example

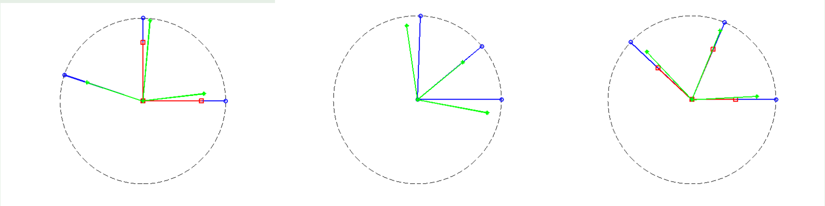

Figure : Frames with 3 vectors in $\mathbb{R}^2$. The original frames are in blue, the frames obtained by scaling (when there exist) are in red, and for comparison the associated canonical tight frames are in green.

Scalable frames in $\mathbb{R}^2$ and $\mathbb{R}^3$

Proposition (G. Kutyniok, F. Philipp, E. K. Tuley, K. O. (2012))

(i) *A frame $\Phi \subset \mathbb{R}^2 \setminus \{0\}$ for $\mathbb{R}^2$ is not scalable if and only if there exists an open quadrant cone which contains all frame vectors of Φ.*

(ii) *A frame $\Phi \subset \mathbb{R}^3 \setminus \{0\}$ for $\mathbb{R}^3$ is not scalable if and only if all frame vectors of Φ are contained in the interior of an elliptical conical surface with vertex 0 and intersecting the corners of a rotated unit cube.*

A geometric characterization of scalable frames

Theorem (G. Kutyniok, F. Philipp, K. Tuley, K.O. (2012))

Let $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N \setminus \{0\}$ *be a frame for* $\mathbb{R}^N$. *Then the following statements are equivalent.*

(i) Φ *is not scalable.*

(ii) *There exists a symmetric* $M \times M$ *matrix* Y *with* $trace(Y) < 0$ *such that* $\langle \varphi_j, Y\varphi_j \rangle \geq 0$ *for all* $j = 1, \ldots, M$.

(iii) *There exists a symmetric* $M \times M$ *matrix* Y *with* $trace(Y) = 0$ *such that* $\langle \varphi_j, Y\varphi_j \rangle > 0$ *for all* $j = 1, \ldots, M$.

Fritz John's Theorem

Theorem (F. John (1948))

Let $K \subset B = B(0,1)$ be a convex body with nonempty interior. There exits a unique ellipsoid $\mathcal{E}_{min}$ of minimal volume containing K.
Moreover, $\mathcal{E}_{min} = B$ if and only if there exist $\{\lambda_k\}_{k=1}^m \subset [0,\infty)$ and $\{u_k\}_{k=1}^m \subset \partial K \cap S^{N-1}$, $m \geq N+1$ such that

(i) $\sum_{k=1}^m \lambda_k u_k = 0$

(ii) $x = \sum_{k=1}^m \lambda_k \langle x, u_k\rangle u_k, \forall x \in \mathbb{R}^N$.

In particular, the points u_k are contact points of K and S^{N-1}.

Frame interpretation of F. John Theorem

Remark

Let $\{u_k\} \subset \partial K \cap S^{N-1}$ *be the contact points of* K *and* S^{N-1}*. The second part of John's theorem can be written:*

$$I_d = \sum_{k=1}^{m} \lambda_k \langle \cdot, u_k \rangle u_k = \sum_{k=1}^{m} \langle \cdot, \sqrt{\lambda_k} u_k \rangle \sqrt{\lambda_k} u_k.$$

So the contact points $\{u_k\}\_k = 1^m$ *form a frame in* $\mathbb{R}^N$*, then we just transformed this frame into an optimally conditioned, i.e., tight frame* $\{\sqrt{\lambda_k} u_k\}_{k=1}^m$*!*

F. John's characterization of scalable frames

Setting

Let $\Phi = \{\varphi_k\}_{k=1}^{M} \subset S^{N-1}$ be a frame for $\mathbb{R}^N$. We apply F. John's theorem to the convex body $K = P_\Phi = conv(\{\pm\varphi_k\}_{k=1}^{M})$. Let $\mathcal{E}_\Phi$ denote the ellipsoid of minimal volume containing P_Φ, and $V_\Phi = Vol(\mathcal{E}_\Phi)/\omega_N$ where ω_N is the volume of the euclidean unit ball.

Theorem (Chen, Kutyniok, Philipp, Wang, K.O. (2014))

Let $\Phi = \{\varphi_k\}_{k=1}^{M} \subset S^{N-1}$ be a frame. Then Φ is scalable if and only if $V_\Phi = 1$. In this case, the ellipsoid $\mathcal{E}_\Phi$ of minimal volume containing $P_\Phi = conv(\{\pm\varphi_k\}_{k=1}^{M})$ is the euclidean unit ball B.

F. John's characterization of scalable frames

Setting

Let $\Phi = \{\varphi_k\}_{k=1}^M \subset S^{N-1}$ be a frame for $\mathbb{R}^N$. We apply F. John's theorem to the convex body $K = P_\Phi = conv(\{\pm\varphi_k\}_{k=1}^M)$. Let $\mathcal{E}_\Phi$ denote the ellipsoid of minimal volume containing P_Φ, and $V_\Phi =$ Vol$(\mathcal{E}_\Phi)/\omega_N$ where ω_N is the volume of the euclidean unit ball.

Theorem (Chen, Kutyniok, Philipp, Wang, K.O. (2014))

Let $\Phi = \{\varphi_k\}_{k=1}^M \subset S^{N-1}$ be a frame. Then Φ is scalable if and only if $V_\Phi = 1$. In this case, the ellipsoid $\mathcal{E}_\Phi$ of minimal volume containing $P_\Phi = conv(\{\pm\varphi_k\}_{k=1}^M)$ is the euclidean unit ball B.

Numerical aspects of F. John's characterization of scalable frames

1. Let $\Phi = \{\varphi_k\}_{k=1}^M \subset S^{N-1}$. What is the cost of computing V_Φ?
2. Khachiyan's barycentric coordinate descent algorithm: $E \supseteq P_\Phi$ with $\mathrm{Vol}(E) \leq (1+\eta)\,\mathrm{Vol}(\text{Minimal ellipsoid}(P_\Phi))$ with a total of $O(M^{3.5}\ln(M\eta^{-1}))$ operations: L. G. Khachiyan (1996).
3. Can be reduced to $O(MN^3\eta^{-1})$ when $N \ll M$: P. Kumar and E. A. Yildirim (2005).
4. Can one find other (algorithmic) methods to optimally condition a frame?
5. What happen when $V_\phi < 1$?

A quadratic programing approach to optimally conditioning frames

Setting

$\Phi = \{\varphi_i\}_{i=1}^M$ *is scalable* $\iff \Phi C \Phi^T = I.$

Let $C_\Phi = \{\Phi C \Phi^T = \sum_{i=1}^M c_i \varphi_i \varphi_i^T : c_i \geq 0\}$ *be the cone generated by* $\{\varphi_i \varphi_i^T\}_{i=1}^M$.

$\Phi = \{\varphi_i\}_{i=1}^M$ *is scalable* $\iff I \in C_\Phi$.

$$D_\Phi := \min_{C \geq 0 \text{ diagonal}} \left\| \Phi C \Phi^T - I \right\|_F$$

A quadratic programing approach to optimally conditioning frames

Setting

$\Phi = \{\varphi_i\}_{i=1}^M$ *is scalable* $\iff \Phi C \Phi^T = I$.
Let $C_\Phi = \{\Phi C \Phi^T = \sum_{i=1}^M c_i \varphi_i \varphi_i^T : c_i \geq 0\}$ *be the cone generated by* $\{\varphi_i \varphi_i^T\}_{i=1}^M$.
$\Phi = \{\varphi_i\}_{i=1}^M$ *is scalable* $\iff I \in C_\Phi$.

$$D_\Phi := \min_{C \geq 0 \text{ diagonal}} \left\| \Phi C \Phi^T - I \right\|_F$$

A quadratic programing approach to optimally conditioning frames

Setting

$\Phi = \{\varphi_i\}_{i=1}^M$ *is scalable* $\iff \Phi C \Phi^T = I.$
Let $C_\Phi = \{\Phi C \Phi^T = \sum_{i=1}^M c_i \varphi_i \varphi_i^T : c_i \geq 0\}$ *be the cone generated by* $\{\varphi_i \varphi_i^T\}_{i=1}^M$.
$\Phi = \{\varphi_i\}_{i=1}^M$ *is scalable* $\iff I \in C_\Phi$.

$$D_\Phi := \min_{C \geq 0 \text{ diagonal}} \left\| \Phi C \Phi^T - I \right\|_F$$

Comparing D_Φ to the frame potential

Proposition (Chen, Kutyniok, Philipp, Wang, K.O. (2014))

(a) Φ *is scalable if and only if* $D_\Phi = 0$.

(b) *If* $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ *is a unit norm frame we have*

$$D_\Phi^2 \leq N - \frac{M^2}{FP(\Phi)},$$

where $FP(\Phi) = \sum_{k,\ell=1}^M |\langle \varphi_k, \varphi_\ell \rangle|^2$.

Remark

D_Φ *can be computed via Quadratic Programming (QP), and is computationally less expansive to compute that* V_Φ.

Comparing the measures of scalability

Theorem (Chen, Kutyniok, Philipp, Wang, K.O. (2014))

Let $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ *is a unit norm frame, then*

$$\frac{N(1-D_\Phi^2)}{N-D_\Phi^2} \leq V_\Phi^{4/N} \leq \frac{N(N-1-D_\Phi^2)}{(N-1)(N-D_\Phi^2)} \leq 1,$$

where the leftmost inequality requires $D_\Phi < 1$. *Consequently,* $V_\Phi \to 1$ *is equivalent to* $D_\Phi \to 0$.

Comparing the measures of scalability

Values of V_Φ and D_Φ for randomly generated frames of M vectors in $\mathbb{R}^4$.

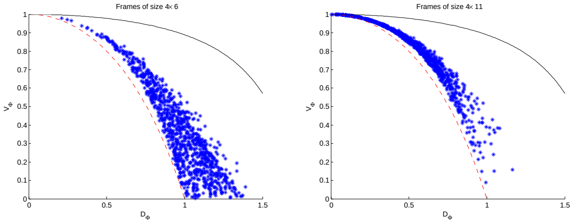

Figure : Relation between V_Φ and D_Φ with $M = 6, 11$. The black line indicates the upper bound in the last theorem, while the red dash line indicates the lower bound.

Comparing the measures of scalability

Values of V_Φ and D_Φ for randomly generated frames of M vectors in $\mathbb{R}^4$.

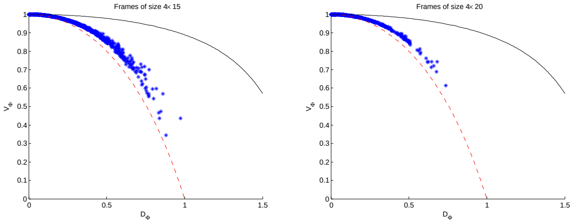

Figure : Relation between V_Φ and D_Φ with $M = 15, 20$. The black line indicates the upper bound in the last theorem, while the red dash line indicates the lower bound.

Probability of a frame to be scalable

Theorem (Chen, Kutyniok, Philipp, Wang, K.O. (2014))

Let $\Phi = \{\varphi_i\}_{i=1}^M \subset \mathbb{R}^N$ *be a frame such that each frame vector* φ_i *is drawn independently and uniformly from* S^{N-1}*. Let* $P_{M,N}$ *be the probability of* Φ *being scalable, then*

(a) $P_{M,N} = 0$*, when* $M < \frac{N(N+1)}{2}$*,*

(b) $P_{M,N} > 0$*, when* $M \geq \frac{N(N+1)}{2}$*, and*

$$g(M,N) \leq P_{M,N} \leq f(M,N),$$

where $\lim_{M\to\infty} f(M,N) = \lim_{M\to\infty} g(M,N) = 1$*.*
Consequently, $\lim_{M\to\infty} P_{M,N} = 1$*.*

Scalable frames: when and how?

Question

Let $\Phi = \{\varphi_k\}_{k=1}^M \subset S^{N-1}$ *be a frame.*

1. V_Φ *and* D_Φ *are ideal measures of scalability.*
2. *If* $V_\Phi = 1$ *(equivalently* $D_\Phi = 0$*) how to find the coefficients needed to make the frame scalable?*
3. *If* $V_\Phi < 1$ *(equivalently* $D_\Phi > 0$*), then* Φ *is not scalable. Can one find* $\{c_k\}_{k=1}^M \subset [0, \infty)$ *such that* $\{c_k\varphi_k\}_{k=1}^M$ *is "almost tight", i.e., its condition number is* $1 + \epsilon$*?*

Scalable frames and Farkas's lemma

Setting

Let $F:\mathbb{R}^N\to\mathbb{R}^d$, $d:=(N-1)(N+2)/2$, *defined by*

$$F(x)=\begin{pmatrix}F_0(x)\\F_1(x)\\\vdots\\F_{N-1}(x)\end{pmatrix}$$

$$F_0(x)=\begin{pmatrix}x_1^2-x_2^2\\x_1^2-x_3^2\\\vdots\\x_1^2-x_N^2\end{pmatrix},\ldots,F_k(x)=\begin{pmatrix}x_kx_{k+1}\\x_kx_{k+2}\\\vdots\\x_kx_N\end{pmatrix}$$

Scalable frames and Farkas's lemma

Theorem (G. Kutyniok, F. Philipp, K.O. (2013))

$\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ is scalable if and only if $F(\Phi)u = 0$ has a nonnegative non trivial solution, where $F(\Phi)$ is the $d \times M$ matrix whose k^{th} row is $F(\varphi_k)$. This is equivalent to 0 being in the relative interior of the convex polytope whose extreme points are $\{F(\varphi_k)\}_{k=1}^M$.

Scalable frames and Farkas's lemma

Lemma (Farkas' Lemma)

For every real $N \times M$-matrix A exactly one of the following cases occurs:

(i) *The system of linear equations $Ax = 0$ has a nontrivial nonnegative solution $x \in \mathbb{R}^M$ (i.e., all components of x are nonnegative and at least one of them is strictly positive.)*

(ii) *There exists $y \in \mathbb{R}^N$ such that $A^T y$ is a vector with all entries strictly positive.*

Scalable frames and Farkas's lemma

Remark

1. *Solving* $F(\Phi)u = 0 : u \geq 0$ *and* $\|u\|_0 = \#\{k : u_k > 0\} = m$ *can be converted into a linear programing.*
2. *Greedy-type algorithm can be used to solve the corresponding LP*
3. *Even when the frame is not scalable one can a "sub-optimally" conditioned frame*
4. *Use of algorithms similar to some introduced by J. Batson, D. Spielman and N. Srivastava for graph sparsification.*

Concluding remarks

1. Scalable frames are just one method for optimally conditioned a frame.
2. Other methods from preconditioning techniques from numerical linear algebra are now being considered.
3. Application of the theory to construction of tight wavelet frames and wavelet filter banks have been done in dimension $N = 1$: Y. Hur and K. O. (2014). Nontrivial and relies on Fejer-Riesz factorization lemma. Extension to $N \geq 2$ very challenging.
4. Connection to graph sparsification.

References

- J. Cahill and X. Chen, *A note on scalable frames* , (2012) arXiv:1301.7292
- X. Chen, K. A. Okoudjou, and R. Wang, *Measures of scalability,* in preparation.
- M. S. Copenhaver, Y. H. Kim, C. Logan, K. Mayfield, S. K. Narayan, and J. Sheperd, *Diagram vectors and tight frame scaling in finite dimensions,* to appear .
- G. Kutyniok, K. A. Okoudjou, F. Philipp, and K. E. Tuley, *Scalable frames*, Linear Algebra Appl., **438** (2013), 2225–2238.
- G. Kutyniok, K. A. Okoudjou, F. Philipp, *Scalable frames and convex geometry,* preprint.
- F. John, *Extremum problems with inequalities as subsidiary conditions,* Studies and Essays Presented to R.

Thank You!

http://www2.math.umd.edu/ okoudjou