Identification of Operators on Elementary Locally Compact Abelian Groups

Gökhan Civan
gcivan@math.umd.edu

Norbert Wiener Center for Harmonic Analysis and Applications
Department of Mathematics
University of Maryland, College Park

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Time-Variant Linear Channels

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- Set $\kappa(x, y) = \tau(x, x - y)$:

$$g \to \int \kappa(\cdot, y) g(y)\, dy$$

Time-Variant Linear Channels (cont.)

- The spreading function:

$$\eta(x,\omega) = \int \kappa(y, y-x) e^{-2\pi y\omega}\, dy$$

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$$g \to \int \eta(x,\omega) M_\omega T_x g\, dx\, d\omega$$

T_x: translation (time delay)
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- g is transformed into a weighted sum of time-frequency shifts of itself.

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  Yes if $\mu(R) \leq 1$
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- Kailtah's conjecture:
  Yes if $\mu(R) \leq 1$
  No if $\mu(R) > 1$
- Bello [Bel69] removed the restriction that R should be a rectangle.

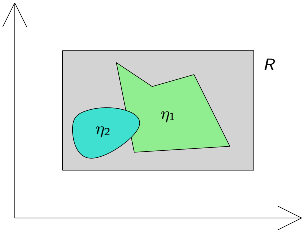
Work of Kozek and Pfander

Theorem (Kozek-Pfander [KP05])

Let R be a rectangle in the time-frequency plane. Consider a family of operators with spreading supports contained in R. If $\mu(R) \leq 1$, then the operator family is identifiable by a Dirac comb

$$g = \sum_{k \in \mathbb{Z}} \delta_{ka}, a > 0.$$

If $\mu(R) > 1$, then there exists no signal which identifies.



Work of Pfander and Walnut

Theorem (Pfander-Walnut [PW06a])

Let S be a set in the time-frequency plane. Consider a family of operators with spreading supports contained in S. If S is compact with $\mu(S) < 1$, then the operator family is identifiable by a periodically weighted Dirac comb

$$g = \sum_{k \in \mathbb{Z}} c_k \delta_{ka}, \;\; c_{k+L} = c_k, a > 0.$$

If S is open with $\mu(S) > 1$, then there exists no signal which identifies.

- Support sets for which identification by a periodically weighted Dirac comb is possible are characterized in [**PW15**] in addition to many other results and reconstruction formulas.
- Many of these results are generalized to arbitrary modulation spaces in [Pfa13b].

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- If e_g is injective, then $\mathcal{O}$ is weakly identifiable by g.
- If e_g is continuous with a bounded inverse (bounded and stable), then $\mathcal{O}$ is strongly identifiable by g.

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- Matrix of e_g: $|\mathbb{A}|^{-1}A(g)_S$
- Immediate observation: If $\mathcal{O}_S$ is identifiable by g, then $|S| \leq |\mathbb{A}|$ ($\mu_{\mathbb{A}\times\widehat{\mathbb{A}}}(S) \leq 1$).

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- $\omega_N = e^{2\pi i/N}$

$$W_N = (\omega_N^{pq})_{p,q=0}^{N} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega_N & \cdots & \omega_N^{N-1} \\ \vdots & \vdots & & \vdots \\ 1 & \omega_N^{N-1} & \cdots & \omega_N^{(N-1)^2} \end{pmatrix}$$

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- $T_k(g) = \mathrm{diag}(g(k), g(k+1), \ldots, g(k-1))$
- $A(g) = (T_0(g)W_N \mid T_1(g)W_N \mid \cdots \mid T_{N-1}(g)W_N)$

Cyclic Case (cont.)

Theorem (Lawrence-Pfander-Walnut [LPW05])

Suppose that N is prime. The product of all $K \times K$ $(1 \leq K \leq N)$ determinants of $A(g)$, interpreted as a polynomial in the indeterminates $g(0), \ldots, g(N-1)$, does not vanish identically.

Theorem (Malikiosis [Mal15])

The product of all $N \times N$ determinants of $A(g)$, interpreted as a polynomial in the indeterminates $g(0), \ldots, g(N-1)$, does not vanish identically.

- Choose g in the complement of the zero set of this polynomial. Then every $N \times N$ minor of $A(g)$ is invertible.
- $|S| \leq N$ implies $\mathcal{O}_S$ is identifiable by g.

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- $\mathbb{A} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
- $g \in \mathbb{C}^{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}$
- $(c_1, c_2, c_3, c_4) = (g(0,0), g(0,1), g(1,0), g(1,1))$

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$$A(g) = \left(\begin{matrix} c_1 & c_1 & c_1 & c_1 & c_2 & c_2 & c_2 & c_2 & \cdots \\ c_2 & -c_2 & c_2 & -c_2 & c_1 & -c_1 & c_1 & -c_1 & \cdots \\ c_3 & c_3 & -c_3 & -c_3 & c_4 & c_4 & -c_4 & -c_4 & \cdots \\ c_4 & -c_4 & -c_4 & c_4 & c_3 & -c_3 & -c_3 & c_3 & \cdots \end{matrix}\right.$$

$$\left.\begin{matrix} \cdots & c_3 & c_3 & c_3 & c_3 & c_4 & c_4 & c_4 & c_4 \\ \cdots & c_4 & -c_4 & c_4 & -c_4 & c_3 & -c_3 & c_3 & -c_3 \\ \cdots & c_1 & c_1 & -c_1 & -c_1 & c_2 & c_2 & -c_2 & -c_2 \\ \cdots & c_2 & -c_2 & -c_2 & c_2 & c_1 & -c_1 & -c_1 & c_1 \end{matrix}\right)$$

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\end{matrix}\right. \\
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- 240 sets $S \subseteq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^{\widehat{}}$ with $|S| = 4$ for which $\mathcal{O}_S$ is not identifiable

Necessary and Sufficient Condition: $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

- $c \in \mathbb{C}^{\mathbb{Z}/2\mathbb{Z}}$

Theorem

Let $S \subseteq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})\widehat{}$. Then $\mathcal{O}_S$ is identifiable by g if and only if (a) the translations of S by $\Gamma \times \Lambda$ are disjoint, and (b) no three of the translations of S by $\Lambda^\perp \times \Gamma^\perp$ have nonempty intersection.

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- Corresponding 4×4 determinants all belong to the list

$$\pm 16 c_0^2 c_1^2, \quad \pm 8 c_0 c_1 (c_0 - c_1)(c_0 + c_1), \quad \pm 8 c_0 c_1 (c_0^2 + c_1^2),$$
$$\pm 4 (c_0 - c_1)^2 (c_0 + c_1)^2, \quad \pm 4 (c_0 - c_1)(c_0 + c_1)(c_0^2 + c_1^2),$$
$$\pm 4 (c_0^2 + c_1^2)^2.$$

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- $\mathcal{O}_S$ is indeed identifiable by g for these sets.

Necessary and Sufficient Condition: $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (cont.)

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- $\mathcal{O}_S$ is indeed identifiable by g for these sets.
- Remaining 4×4 determinants are all zero.
- For the remaining sets, $\mathcal{O}_S$ is indeed not identifiable by g.

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- ELCA group G:

$$G = \mathbb{R}^d \times \mathbb{T}^{d'} \times \mathbb{Z}^{d''} \times \mathbb{A}$$

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  $e_g : \mathcal{O}^{\infty,1}(G) \to M^1(G) \subseteq L^2(G)$
  $e_g|S : \mathcal{O}^{\infty,1}(G)|S \to L^2(G)$

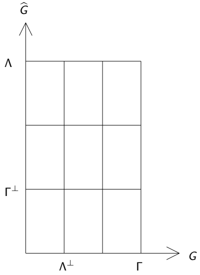
Zak Transform and Quasi-Periodization

- Zak transform of $f \in M^1(G)$:

$$Z_\Gamma f(a, \hat{a}) = \sum_{w \in \Gamma} f(a+w)(-w, \hat{a})$$

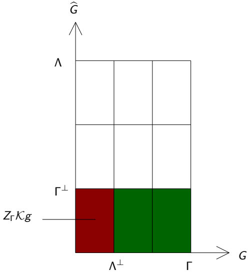
- Quasi-periodization of $\eta \in M^1(G \times \widehat{G})$:

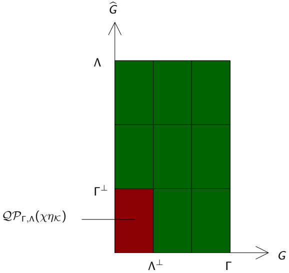
$$\mathcal{QP}_{\Gamma,\Lambda}\eta(a, \hat{a}) = \sum_{w \in \Gamma} \sum_{\upsilon \in \Lambda} \eta(a+w, \hat{a}+\upsilon)(-w, \hat{a})$$



Zak Transform, Quasi-Periodization, and Operators

- $\mathcal{K} \in \mathcal{O}^{\infty,1}(G)$





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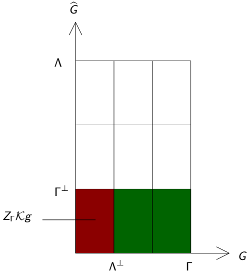

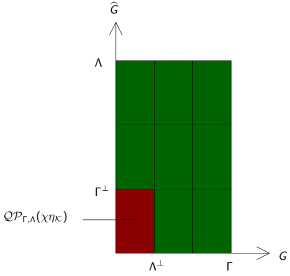

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- $g = \sum_{w \in \Gamma} T_w \delta_G$

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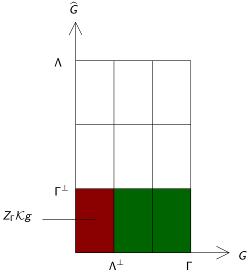

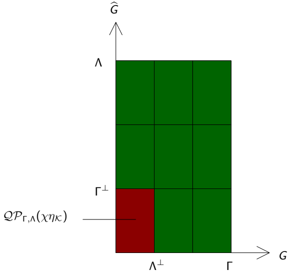

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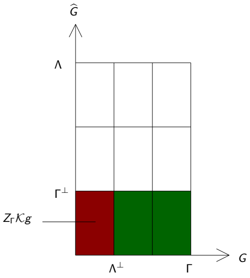

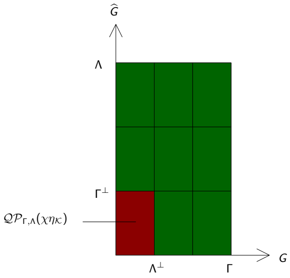

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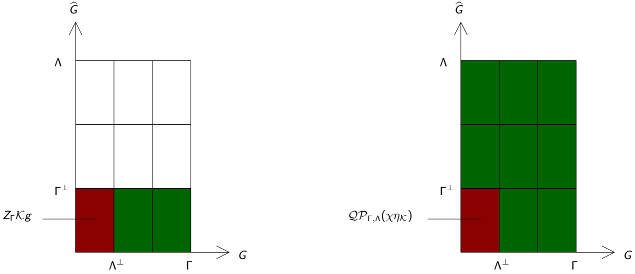

Sufficient Conditions for Operator Identification

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- $S \subseteq G \times \widehat{G}$ open

$$\sum_{k \in \Gamma} \sum_{\ell \in \Lambda} \mathbb{1}_{S+(k,\ell)} \leq 1 \tag{1}$$

and

$$\sum_{\ell^{\perp} \in \Lambda^{\perp}} \sum_{k^{\perp} \in \Gamma^{\perp}} \mathbb{1}_{S+(\ell^{\perp},k^{\perp})} \leq |\Lambda^{\perp}/\Gamma| \tag{2}$$

Sufficient Conditions for Operator Identification (cont.)

Theorem (generalizing [PW15])

The following statements are equivalent:

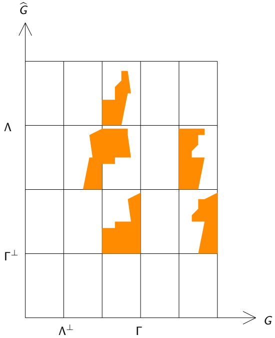
1. (1) *and* (2) *hold pointwise everywhere.*
2. $\mathcal{O}^{\infty,1}(G)|S$ *is strongly identifiable by* g.
3. $\mathcal{O}^{\infty,1}(G)|S$ *is weakly identifiable by* g.

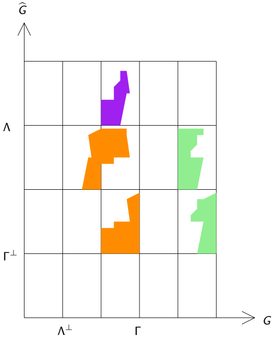
Corollary (generalizing [PW06a, Theorem 3.1])

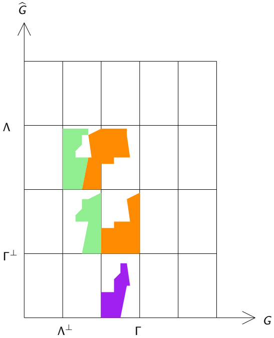
Suppose that G *has at most one finite cyclic summand. Let* $S \subseteq G \times \widehat{G}$ *be compact with* $\mu_{G \times \widehat{G}}(S) < 1$. *Then* $\mathcal{O}^{\infty,1}(G)|S$ *is strongly identifiable.*

Description of (1)

$$\sum_{k\in\Gamma}\sum_{\ell\in\Lambda}\mathbb{1}_{S+(k,\ell)} \le 1$$

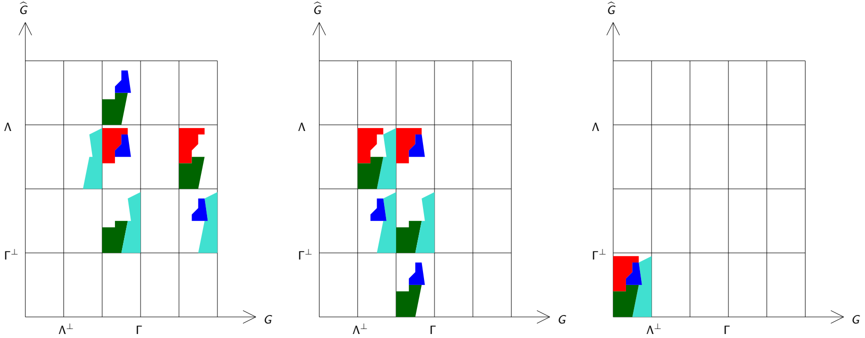






Description of (2)

$$\sum_{\ell^\perp \in \Lambda^\perp} \sum_{k^\perp \in \Gamma^\perp} \mathbb{1}_{S+(\ell^\perp, k^\perp)} \leq |\Lambda^\perp / \Gamma|$$



Proof of Theorem *(a)* $\Rightarrow$ *(b)*

- $J \subseteq (\Lambda^{\perp}/\Gamma) \times (\Gamma^{\perp}/\Lambda)$ with $|J| \leq |\Lambda^{\perp}/\Gamma|$

Proof of Theorem *(a)* ⇒ *(b)*

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- $\mathcal{K} \in \mathcal{O}^{\infty,1}(G)|S$

$$\mathbf{Z}_\Gamma \mathcal{K} g = \mu_{\widehat{G}}(D^\perp) A(c)_J \boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda,J}$$

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- $$\mu_{\widehat{G}}(D^\perp)^2 a_J^2 \|\boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda,J}\|_2^2 \leq \|\mathbf{Z}_\Gamma \mathcal{K} g\|_2^2 \leq \mu_{\widehat{G}}(D^\perp)^2 b_J^2 \|\boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda,J}\|_2^2$$

  on V_J

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  on V_J
- $$\mu_{\widehat{G}}(D^\perp)^2 a^2 \|\boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda}\|_2^2 \leq \|\mathbf{Z}_\Gamma \mathcal{K} g\|_2^2 \leq \mu_{\widehat{G}}(D^\perp)^2 b^2 \|\boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda}\|_2^2$$

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- $$\mu_{\widehat{G}}(D^\perp)^2 a_J^2 \|\boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda,J}\|_2^2 \leq \|\mathbf{Z}_\Gamma \mathcal{K} g\|_2^2 \leq \mu_{\widehat{G}}(D^\perp)^2 b_J^2 \|\boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda,J}\|_2^2$$
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- $$\mu_{\widehat{G}}(D^\perp)^2 a^2 \|\boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda}\|_2^2 \leq \|\mathbf{Z}_\Gamma \mathcal{K} g\|_2^2 \leq \mu_{\widehat{G}}(D^\perp)^2 b^2 \|\boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda}\|_2^2$$
- $$\mu_{\widehat{G}}(D^\perp) a^2 \|\eta_{\mathcal{K}}\|_2^2 \leq \|e_g \mathcal{K}\|_2^2 \leq \mu_{\widehat{G}}(D^\perp) b^2 \|\eta_{\mathcal{K}}\|_2^2$$

A Duality Principle

- $\mathcal{K} \in \mathcal{O}^{\infty,1}(G)$

Theorem

$\mathcal{O}^{\infty,1}(G)|S$ is strongly identifiable by g if and only if $\mathcal{O}^{\infty,1}(\widehat{G})|S_{\mathcal{F}}$ is strongly identifiable by $\hat{g}$.

A Duality Principle

- $\mathcal{K} \in \mathcal{O}^{\infty,1}(G)$
- $\mathcal{K}_{\mathcal{F}} \in \mathcal{O}^{\infty,1}(\widehat{G})$
  $\eta_{\mathcal{K}_{\mathcal{F}}}(\hat{a}, a) = (-a, \hat{a})\eta_{\mathcal{K}}(-a, \hat{a})$

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  $S_{\mathcal{F}} = \{(\hat{a}, a) \in \widehat{G} \times G : (-a, \hat{a}) \in S\}$
- $$\begin{array}{ccc} \mathcal{O}^{\infty,1}(G)|S & \xrightarrow{e_g} & L^2(G) \\ {\scriptstyle \mathcal{K}\to\mathcal{K}_{\mathcal{F}}}\big\downarrow & & {\scriptstyle \mathcal{F}}\big\downarrow \\ \mathcal{O}^{\infty,1}(\widehat{G})|S_{\mathcal{F}} & \xrightarrow{e_{\hat{g}}} & L^2(\widehat{G}) \end{array}$$

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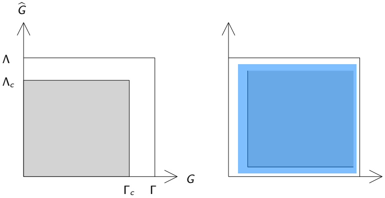
$\mathcal{O}^{\infty,1}(G)|S$ is strongly identifiable by g if and only if $\mathcal{O}^{\infty,1}(\widehat{G})|S_{\mathcal{F}}$ is strongly identifiable by $\hat{g}$.

A Riesz Basis of Operators



Theorem

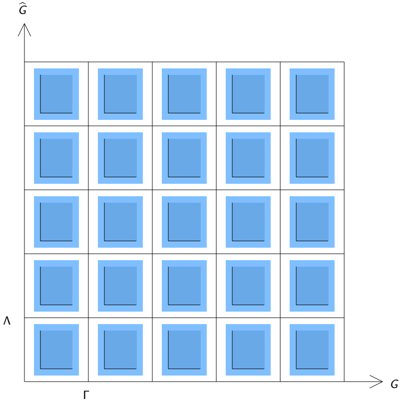
$$\{M_{\upsilon+w_c^\perp} T_{-\upsilon_c^\perp} \mathcal{P} T_{w+\upsilon_c^\perp} M_{-w_c^\perp}\}_{(w,\upsilon,w_c^\perp,\upsilon_c^\perp)\in\Gamma\times\Lambda\times\Gamma_c^\perp\times\Lambda_c^\perp}$$

is a Riesz basis for its closed linear span in $\mathcal{O}^2(G)$.

- $\eta_{M_{\upsilon+w_c^\perp} T_{-\upsilon_c^\perp} \mathcal{P} T_{w+\upsilon_c^\perp} M_{-w_c^\perp}} = (-\upsilon_c^\perp, \upsilon) M_{(w_c^\perp, \upsilon_c^\perp)} T_{(w,\upsilon)} \eta_{\mathcal{P}}$
- $U : \ell_c(\Gamma \times \Lambda \times \Gamma_c^\perp \times \Lambda_c^\perp) \to \mathcal{O}^{\infty,1}(G)$

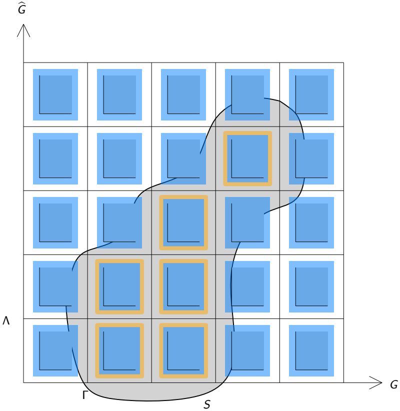
A Riesz Basis of Operators (cont.)

$$\eta M_{\upsilon + w_c^\perp} T_{-\upsilon_c^\perp} \mathcal{P} T_{w+\upsilon_c^\perp} M_{-w_c^\perp} = (-\upsilon_c^\perp, \upsilon) M_{(w_c^\perp, \upsilon_c^\perp)} T_{(w,\upsilon)} \eta \mathcal{P}$$



Restricting the Identification Problem

- $J \subseteq \Gamma \times \Lambda$ finite



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- $J \subseteq \Gamma \times \Lambda$ finite
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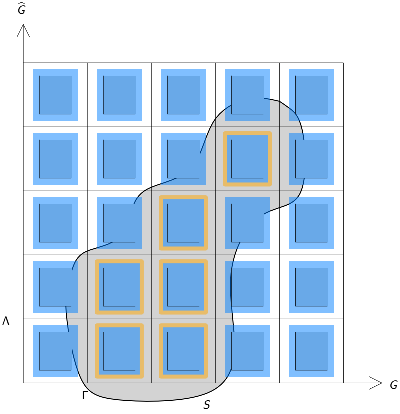
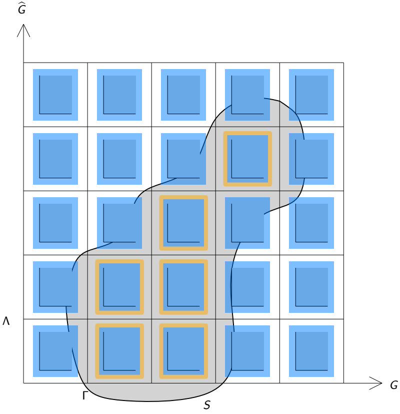


Restricting the Identification Problem

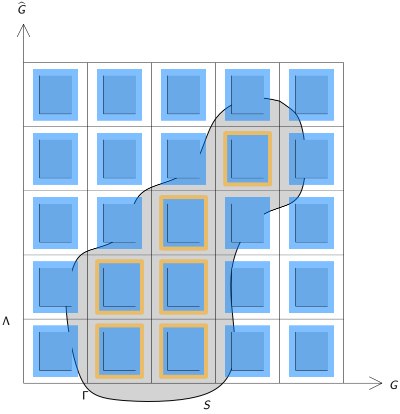
- $J \subseteq \Gamma \times \Lambda$ finite
- $\mathcal{V}_J$: image of $U \circ i_J$
- Arrange $\mathcal{V}_J \subseteq \mathcal{O}^{\infty,1}(G)|S$.



Restricting the Identification Problem

- $J \subseteq \Gamma \times \Lambda$ finite
- $\mathcal{V}_J$: image of $U \circ i_J$
- Arrange $\mathcal{V}_J \subseteq \mathcal{O}^{\infty,1}(G)|S$.
- Restrict to $\mathcal{V}_J$:

$$e_g \circ U \circ i_J = e_g|S \circ U \circ i_J$$



Simplifying the RHS

- $V : L^2(G) \to \ell^2(\mathbb{Z})$
  $V \circ e_g|S \circ U \circ i_J$

Lemma ([KP05, Lemma 3.4])

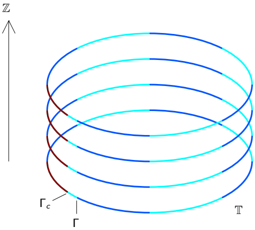
Let $g \in M^\infty(G)$. There exists a nonnegative continuous function r on G, decreasing faster than any polynomial, such that $|\mathcal{P}M_{\hat{b}}T_b g| \leq r$. There exists a nonnegative continuous function $r_{\mathcal{F}}$ on $\widehat{G}$, decreasing faster than any polynomial, such that $|(\mathcal{P}M_{\hat{b}}T_b g)^\frown| \leq r_{\mathcal{F}}$.

Proposition ([Pfa08, Theorem 2.1])

Let $A : \ell_c(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ be a (not necessarily bounded) linear map. Let $(a_{k',k})_{k',k\in\mathbb{Z}^d}$ be the matrix representation of A with respect to the orthonormal bases $\{T_{k'}\delta_{\mathbb{Z}^d}\}_{k'\in\mathbb{Z}^d}$ and $\{T_k\delta_{\mathbb{Z}^d}\}_{k\in\mathbb{Z}^d}$. Let $\tilde{r}$ be a nonnegative Borel measurable function on $\mathbb{R}$, decreasing faster than any polynomial. Let $\lambda > 1$. Suppose that $|a_{k',k}| \leq \tilde{r}(\|\lambda k' - k\|_\infty)$. In this case, there does not exist a bounded linear map $B : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ with $BA = I$.

Example: The Circle

- $L > K$
  $D = [0, 1/K)$, $D_c = [0, 1/L)$



Theorem

Let $S \subseteq \mathbb{T} \times \mathbb{Z}$ be open with $\mu_{\mathbb{T}\times\mathbb{Z}}(S) > 1$. There exists no $g \in M^\infty(\mathbb{T})$ for which $e_g|S$ is stable.

Corollary

Let $S \subseteq \mathbb{Z} \times \mathbb{T}$ be open with $\mu_{\mathbb{Z}\times\mathbb{T}}(S) > 1$. There exists no $g \in \ell^\infty(\mathbb{Z})$ for which $e_g|S$ is stable.

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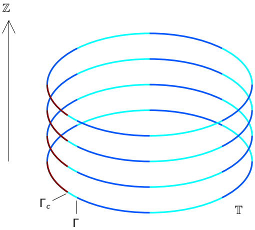

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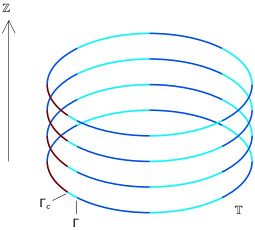

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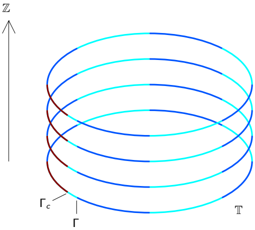

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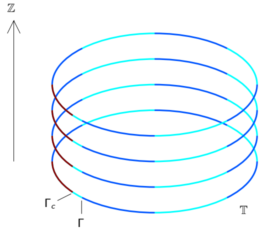

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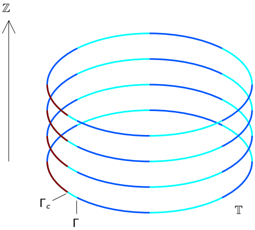
- $J \subseteq \Gamma \times \Lambda$
  $\lambda = |J|/L > 1$
- $|a_{\xi,(k_j,p_j,q)}| \leq \tilde{r}(\lambda\xi - (q|J| + j))$
- $e_g \circ U \circ i_J$ is not stable



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Product Groups

- $\eta = \eta_1 \otimes \eta_2 \Rightarrow \mathcal{K}(g_1 \otimes g_2) = (\mathcal{K}_1 g_1) \otimes (\mathcal{K}_2 g_2)$

Theorem

Suppose that G_1 has the finely tuned overspreading property. Let $S \subseteq G_1 \times G_2 \times \widehat{G}_1 \times \widehat{G}_2$ be open. Suppose that there exists $(a_2, \hat{a}_2) \in G_2 \times \widehat{G}_2$ such that $\mu_{G_1 \times \widehat{G}_1}(S_{(a_2,\hat{a}_2)}) > 1$, where

$$S_{(a_2,\hat{a}_2)} = \{(a_1, \hat{a}_1) \in G_1 \times \widehat{G}_1 : (a_1, a_2, \hat{a}_1, \hat{a}_2) \in S\}$$

In this case, there exist no $g_1 \in M^\infty(G_1)$ and $g_2 \in M^\infty(G_2)$ for which $e_{g_1 \otimes g_2}|S$ is stable.

Product Groups

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- $U : \ell_c(\Gamma_1 \times \Gamma_2 \times \Lambda_1 \times \Lambda_2 \times \Gamma_{1,c}^{\perp} \times \Gamma_{2,c}^{\perp} \times \Lambda_{1,c}^{\perp} \times \Lambda_{2,c}^{\perp}) \to \mathcal{O}^{\infty,1}(G_1 \times G_2)$

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- $$\begin{aligned} &U(\sigma_1 \otimes T_{(w_2,\upsilon_2,w_{2,c}^{\perp},\upsilon_{2,c}^{\perp})}\delta_{\Gamma_2 \times \Lambda_2 \times \Gamma_{2,c}^{\perp} \times \Lambda_{2,c}^{\perp}})(g_1 \otimes g_2) \\ =&(U_1 \sigma 1) g_1 \otimes (U_2 T_{(w_2,\upsilon_2,w_{2,c}^{\perp},\upsilon_{2,c}^{\perp})}\delta_{\Gamma_2 \times \Lambda_2 \times \Gamma_{2,c}^{\perp} \times \Lambda_{2,c}^{\perp}}) g_2 \end{aligned}$$

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Suppose that G_1 has the finely tuned overspreading property. Let $S \subseteq G_1 \times G_2 \times \widehat{G}_1 \times \widehat{G}_2$ be open. Suppose that there exists $(a_2, \hat{a}_2) \in G_2 \times \widehat{G}_2$ such that $\mu_{G_1 \times \widehat{G}_1}(S_{(a_2,\hat{a}_2)}) > 1$, where

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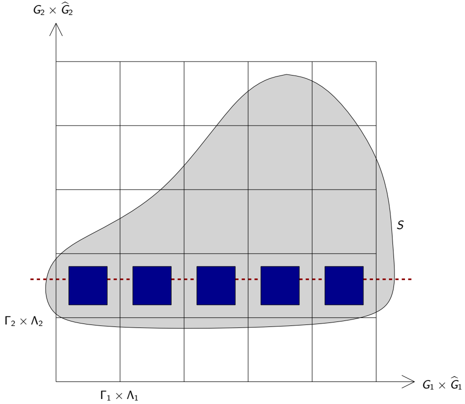
Product Groups (cont.)



Further Questions

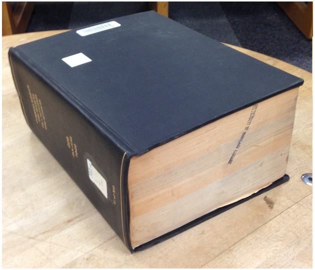
- The underspread condition is necessary for operator identification on $\mathbb{R}$, $\mathbb{T}$, $\mathbb{Z}$, and $\mathbb{A}$ individually. What about in general?

Conjecture

Let G be an arbitrary ELCA group. Let $S \subseteq G \times \widehat{G}$ be open with $\mu_{G\times\widehat{G}}(S) > 1$. There exists no $g \in M^\infty(G)$ for which $e_g|S$ is stable.

- Explicit construction of vectors $c \in \mathbb{C}^{\mathbb{Z}/N\mathbb{Z}}$ such that $A(c)$ is full spark
- Bounds on the Frobenius norms of the $N \times N$ minors and their inverses

Thank You



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