Graph theoretic uncertainty principles

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Uncertainty principles – 1

The Heisenberg uncertainty principle inequality is

$$\forall f \in L^2(\mathbb{R}), \quad \|f\|^2_{L^2(\mathbb{R})} \leq 4\pi \, \|t\, f(t)\|_{L^2(\mathbb{R})} \left\|\gamma\, \hat{f}(\gamma)\right\|_{L^2(\widehat{\mathbb{R}})}.$$

Additively, we have

$$\forall f \in L^2(\mathbb{R}), \quad \|f\|^2_{L^2(\mathbb{R})} \leq 2\pi \left(\|t\, f(t)\|^2_{L^2(\mathbb{R})} + \left\|\gamma\, \hat{f}(\gamma)\right\|^2_{L^2(\widehat{\mathbb{R}})} \right).$$

Equivalently, for $f \in \mathcal{S}(\mathbb{R})$,

$$\|f\|^2_{L^2(\mathbb{R})} \leq \left\|\widehat{f'}\right\|^2_{L^2(\widehat{\mathbb{R}})} + \|f'\|^2_{L^2(\mathbb{R})}.$$

We shall extend this inequality to graphs.

Uncertainty principles – 2

- In signal processing, uncertainty principles dictate the trade off between high spectral and high temporal accuracy, establishing limits on the extent to which the "instantaneous frequency" of a signal can be measured (Gabor, 1946)
- Weighted, Euclidean, LCAG, non-L^2 uncertainty principles, proved by Fourier weighted norm inequalities, e.g., Plancherel, generalizations of Hardy's inequality, e.g., integration by parts, and Hölder (alas).
- DFT: Chebatorov, Grünbaum, Donoho and Stark,Tao.
- Generalize the latter to graphs.

Graph theory – background

- Problem: propose, prove, and understand uncertainty principle inequalities for graphs, see A. Agaskar and Y. M. Lu on *A spectral graph uncertainty principle*
- Generally: There is no obvious solution because of the loss on general graphs of the cyclic structure associated with the DFT.
- Locally: Radar/Lidar data analysis at NWC uses non-linear spectral kernel methods, with *essential* graph theoretic components for dimension reduction and remote sensing.

Graph theory – definition

Definition

A *graph* is $G = \{V, \mathbf{E} \subseteq V \times V, w\}$ consisting of a set V called vertices, a set $\mathbf{E}$ called edges, and a weight function

$$w : V \times V \longrightarrow [0, \infty).$$

Write $V = \{v_j\}_{j=0}^{N-1}$ and keep the ordering fixed, but arbitrary.

Graph theory – assumptions

- For any $(v_i, v_j) \in V \times V$ we have

$$w(v_i, v_j) = \begin{cases} 0 & \text{if } (v_i, v_j) \in \mathbf{E}^c \\ c > 0 & \text{if } (v_i, v_j) \in \mathbf{E}. \end{cases}$$

- G is undirected, i.e., $w(v_i, v_j) = w(v_j, v_i)$.
- $w(v_i, v_i) = 0$, i.e., G has no loops.
- G is connected, i.e., given any v_i and v_j, there exists at most one edge between them, and there exists a sequence of vertices $\{v_k\}, k = 0, ..., d \leq |V| = N$, such that

$$(v_i, v_0), (v_0, v_1), ..., (v_d, v_j) \in \mathbf{E}.$$

- G is *unit weighted* if w takes only the values 0 and 1.

Graph Laplacian

- $N \times N$ symmetric *adjacency matrix*, A, for G :

$$A = (A_{ij}) = (w(v_i, v_j)).$$

- The *degree matrix*, D, is the $N \times N$ diagonal matrix,

$$D = diag\left(\sum_{j=0}^{N-1} A_{0j}, \sum_{j=0}^{N-1} A_{1j}, \cdots, \sum_{j=0}^{N-1} A_{(N-1)j}\right).$$

- The *graph Laplacian*,

$$L = D - A,$$

  is the $N \times N$ symmetric, positive semi-definite matrix, with real ordered eigenvalues $0 = \lambda_0 \leq \ldots \leq \lambda_{N-1}$ and orthonormal eigenbasis, $\{\chi_j\}_{j=0}^{N-1}$, for $\mathbb{R}^N$.

Graph Fourier transform

- Formally, the Fourier transform $\hat{f}$ at γ of f defined on $\mathbb{R}$ is the inner product of f with the complex exponentials, that are the eigenfunctions of the Laplacian operator $\frac{d^2}{dt^2}$ on $\mathbb{R}$.
- Thus, define the *graph Fourier transform*, $\widehat{f}$, of $f \in \ell^2(G)$ in the graph Laplacian eigenbasis:

$$\widehat{f}[j] = \langle \chi_j, f \rangle, \quad j = 0, \ldots, N-1.$$

If

$$\chi = [\chi_0, \chi_1, ..., \chi_{N-1}],$$

then $\widehat{f} = \chi^* f$, and, since χ is unitary, we have the *inversion formula*:

$$f = \chi \chi^* f = \chi \hat{f}.$$

Difference operator for graphs

The *difference operator*,

$$D_r : \ell^2(G) \longrightarrow \mathbb{R}^{|\mathbf{E}|},$$

with coordinate values representing the change in f over each edge, is defined by

$$(D_r f)[k] = (f[j] - f[i])\,(w(e_k))^{1/2},$$

where $e_k = (v_j, v_i)$ and $j < i$.

- D_r can be defined by the *incidence matrix* of G.
- If G is a unit weighted circulant graph, then D_r is the intuitive difference operator of Lammers and Maeser.

Difference uncertainty principle for graphs

Theorem

Let G be a connected and undirected graph. Then,

$$\forall f \in \ell^2(G), \quad 0 < \tilde{\lambda}_0 \|f\|^2 \leq \|D_r f\|^2 + \left\|D_r \hat{f}\right\|^2 \leq \tilde{\lambda}_{N-1} \|f\|^2,$$

where

$$\Delta = \mathrm{diag}\{\lambda_0, \ldots, \lambda_{N-1}\}$$

and where $0 < \tilde{\lambda}_0 \leq \tilde{\lambda}_1 \leq \ldots \leq \tilde{\lambda}_{N-1}$ are the eigenvalues of $L + \Delta$. The bounds are sharp.

Frame difference uncertainty principle for graphs

$\{e_j\}_{j=0}^{N-1} \subseteq \mathbb{C}^d$ is a *frame* for $\mathbb{C}^d$ if

$$\exists\, 0 < A \leq B \,\text{such that}\, \forall f \in \mathbb{C}^d, \quad 0 < A\left\|f\right\|^2 \leq \sum_{j=0}^{N-1} |\langle f, e_j\rangle|^2 \leq B\left\|f\right\|^2.$$

- If $A = B = 1$ then the frame is a *Parseval frame*.
- Define the $d \times N$ matrix $E = [e_0, e_1, \ldots, e_{N-1}]$, where $\{e_j\}_{j=0}^{N-1}$ is a Parseval frame for $\mathbb{C}^d$. Then $EE^* = I_{d\times d}$.

Theorem

Let G be a connected and undirected graph. Then, for every $d \times N$ Parseval frame E,

$$\sum_{j=0}^{d-1} \tilde{\lambda}_j \leq \|D_r \chi^* E^*\|_{fr}^2 + \|D_r E^*\|_{fr}^2 \leq \sum_{j=N-d}^{N-1} \tilde{\lambda}_j.$$

The bounds are sharp.

Feasibility region

The difference operator *feasibility region FR* is

$$FR = \{(x, y) : \exists f \in \ell^2(G),\ \|f\| = 1,\ \text{such that}\ \|D_r f\|^2 = x \text{ and } \left\|D_r \widehat{f}\right\|^2 = y\}.$$

Theorem

a. FR is a closed subset subset of $[0, \lambda_{N-1}] \times [0, \lambda_{N-1}]$, where λ_{N-1} is the maximum eigenvalue of the Laplacian L.
b. $(\frac{1}{N}\sum_{j=0}^{N-1} \lambda_j, 0)$ and $(0, L_{0,0})$ are the only points of FR on the axes.
c. FR is in the half plane defined by $x + y \geq \tilde{\lambda}_0 > 0$ with equality if and only if $\hat{f}$ is in the eigenspace associated with $\tilde{\lambda}_0$.
d. If $N \geq 3$, then FR is a convex region.

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Complete graph

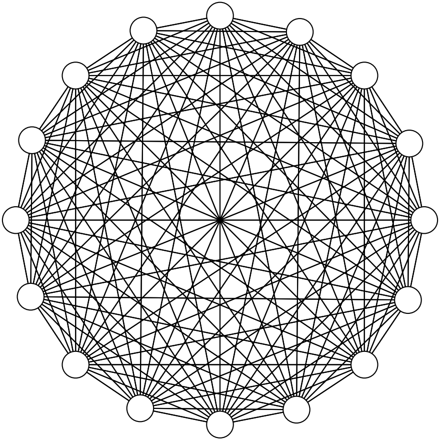

Figure : A unit weighted complete graph with 16 vertices.

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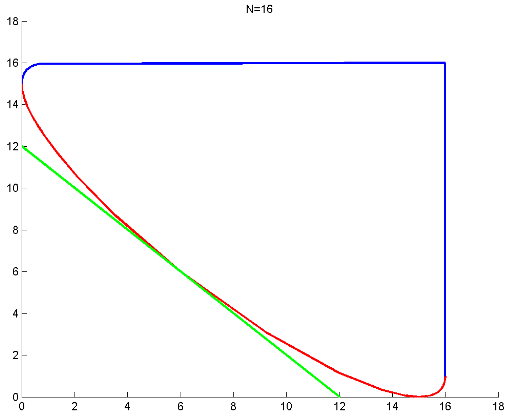
Feasibility region



Difference uncertainty curve

The *difference uncertainty curve* ω is the lower boundary of *FR* defined as

$$\forall x \in [0, \lambda_{N-1}], \quad \omega(x) = \inf_{g \in \ell^2(G)} \langle g, Lg \rangle$$

$$\text{subject to } \langle g, \Delta g \rangle = x.$$

Given $x \in [0, \lambda_{N-1}]$. $g_x \in \ell^2(G)$ *attains the difference uncertainty curve* at x if, for all g for which $\langle g, \Delta g \rangle = x$, we have

$$\langle g_x, Lg_x \rangle \leq \langle g, Lg \rangle.$$

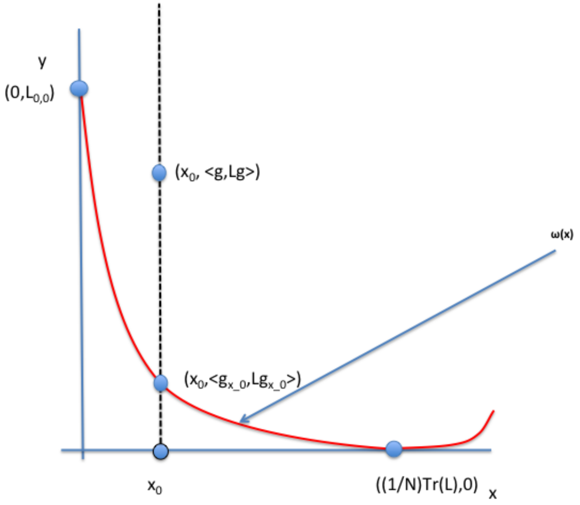


Figure : The difference uncertainty curve (red) for a connected graph G

Uncertainty curve theorem

Theorem

A unit normed function $f \in \ell^2(G)$, with $\|D_r f\|^2 = x \in (0, \lambda_{N-1})$, achieves the uncertainty curve at x if and only if $\hat{f}$ is a nonzero eigenfunction for $K(\alpha) = L - \alpha\Delta$ associated with the minimal eigenvalue of $K(\alpha)$, where $\alpha \in (-\infty, \infty)$.

Uncertainty principle problem and comparison

- Lammers and Maeser, Grünbaum, Agaskar and Lu.
- The Agaskar and Lu problem.
- Critical comparison between the graph theoretical feasibility region and the comparable Bell Labs uncertainty principle region.

Thats all folks!