**On Constant Amplitude Zero Autocorrelation Sequences of length $p$**

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**Introduction**

Define a function $u: \mathbb{Z}_n \to \mathbb{C}$. We define the inner product

$$\langle u(k), u(m) \rangle = u(k)\overline{u(m)}.$$  

We say that $u$ is unimodular if

$$\langle u(k), u(k) \rangle^{1/2} = 1.$$  

We define the autocorrelation $A_u(m)$ of $u$ for $m \in \mathbb{Z}_n$ as

$$A_u(m) = \frac{1}{n} \sum_{k=0}^{n-1} \langle u(m+k), u(k) \rangle.$$  

We say that $u$ is a CAZAC (Constant Amplitude Zero Autocorrelation) sequence of length $d$ if it is unimodular and if it satisfies $A_u(m) = 0$ for all $m \in \mathbb{Z}_n \setminus \{0\}$. CAZAC sequences are defined to be equivalent under rotation, translation, decimation, linear frequency modulation, and conjugation.

**Hadamard Matrices**

As defined in Benedetto and Datta, a complex Hadamard matrix is a square matrix with unimodular entries and mutually orthogonal rows. Benedetto and Datta further showed there is a 1-1 correspondence between CAZAC sequences and circulant Hadamard matrices.

$$F_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \\ \end{pmatrix},$$

where $\omega = e^{2\pi i/3}$ and equivalent means that they can be written as $D_1P_3P_3P_2D_2$ where $P_1$ and $P_2$ are permutation matrices and $D_1$ and $D_2$ are diagonal unitary matrices.

For $5 \times 5$ matrices, using some results by Haagerup, all $5 \times 5$ Hadamard matrices are equivalent to

$$F_5 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 \\ 1 & \omega^2 & \omega & \omega^3 & \omega^4 \\ 1 & \omega^3 & \omega^2 & \omega & \omega^4 \\ 1 & \omega^4 & \omega^3 & \omega^2 & \omega \\ \end{pmatrix},$$

where $\omega = e^{2\pi i/5}$.

**Previous Results, Ctd.**

Letting

$$A = \begin{pmatrix} e^{ia} & 0 & 0 \\ 0 & e^{ib} & 0 \\ 0 & 0 & e^{ic} \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{ix} & 0 \\ 0 & 0 & e^{iy} \end{pmatrix},$$

and

$$F'_{d} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \\ \end{pmatrix},$$

where $a, b, c, x, y \in [0, 2\pi)$ and $\omega = e^{2\pi i/3}$, Bruzda, Tadej, and Życzkowski further claim that the set of $3 \times 3$ Hadamard matrices is

$$\{A \cdot F_3 \cdot B \} \cup \{A \cdot F'_3 \cdot B \}.$$  

**Results**

Using the first form in the characterization of $3 \times 3$ Hadamard matrices by Bruzda et al., and imposing conditions to make the matrix circulant, we obtain the following equations, all mod $2\pi$:

$$a = \frac{2\pi}{3} + b + x = \frac{2\pi}{3} + c + y$$

$$c = (a + x) = \frac{2\pi}{3} + b + y$$

$$a + y = b = \frac{2\pi}{3} + c + x$$

We simplify this further to the following equations (still mod $2\pi$):

$$a = \frac{2\pi}{3} + b + x$$

$$c = \frac{2\pi}{3} + b + 2x$$

$$y = \frac{\pi}{3} + c + x - a = \frac{4\pi}{3} (\frac{2\pi}{3} + b + 2x) +$$

$$x = (\frac{2\pi}{3} + b + x) = 4\pi + 2x +$$

$$b = c + y - x = (\frac{2\pi}{3} + b + 2x) + (\frac{4\pi}{3} + 2x) - x = 2\pi + b + 3x.$$  

**Discussion**

Surprisingly, two of the first rows of the matrices were not CAZACs. One would also expect there to be 6 CAZACs of length 3, based on work by Haagerup. A similar approach makes sense to find length 5 CAZACs. Bruzda et al. do not have a special simplified characterization for $5 \times 5$ Hadamard matrices, so the work is slightly more difficult. It should be noted that in general, it is not known how many CAZACs of length $p$ there are for arbitrary $p$, although Haagerup showed that $\binom{2p}{p}$ is an upper bound on this number, and thus it is finite.