Frames and some algebraic forays

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Outline

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Frames

- Let H be a separable Hilbert space, e.g., $H = L^2(\mathbb{R}^d)$, $\mathbb{R}^d$, or $\mathbb{C}^d$.
- $F = \{x_n\} \subseteq H$ is a *frame* for H if

$$\exists\, A, B > 0 \text{ such that } \forall\, x \in H, \quad A\|x\|^2 \leq \sum |\langle x, x_n\rangle|^2 \leq B\|x\|^2.$$

Theorem

If $F = \{x_n\} \subseteq H$ is a frame for H then

$$\forall x \in H, \quad x = \sum \langle x, S^{-1}x_n\rangle x_n = \sum \langle x, x_n\rangle S^{-1}x_n,$$

where $S : H \to H$, $x \mapsto \sum \langle x, x_n\rangle x_n$ is well-defined.

- Frames are a natural tool for dealing with numerical stability, overcompleteness, noise reduction, and robust representation problems.

Outline

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Ambiguity function and STFT

- Woodward's (1953) *narrow band cross-correlation ambiguity function* of v, w defined on $\mathbb{R}^d$:

$$A(v,w)(t,\gamma) = \int v(s+t)\overline{w(s)}e^{-2\pi i s\cdot\gamma}ds.$$

- The *STFT* of v : $V_w v(t,\gamma) = \int v(x)\overline{w(x-t)}e^{-2\pi i x\cdot\gamma}dx.$
- $A(v,w)(t,\gamma) = e^{2\pi i t\cdot\gamma}V_w v(t,\gamma).$
- The *narrow band ambiguity function* $A(v)$ of v :

$$A(v)(t,\gamma) = A(v,v)(t,\gamma) = \int v(s+t)\overline{v(s)}e^{-2\pi i s\cdot\gamma}ds$$

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The discrete periodic ambiguity function

- Given $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$.
- The *discrete periodic ambiguity function*,

$$A(u) : \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C},$$

  of u is

$$A(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1} u[m+k]\overline{u[k]}e^{-2\pi ikn/N}.$$

- u is *Constant Amplitude Zero Autocorrelation (CAZAC)* if

$$\forall m \in \mathbb{Z}/N\mathbb{Z}, \quad |u[m]| = 1, \text{ (CA)}$$

  and

$$\forall m \in \mathbb{Z}/N\mathbb{Z} \setminus \{0\}, \quad A(u)(m,0) = 0. \text{ (ZAC)}$$

Björck CAZAC discrete periodic ambiguity function

Let $A(b_p)$ be the Björck CAZAC discrete periodic ambiguity function defined on $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

Theorem (J. and R. Benedetto and J. Woodworth [2])

$$|A(b_p)(m,n)| \leq \frac{2}{\sqrt{p}} + \frac{4}{p}$$

for all $(m,n) \in \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \setminus (0,0)$.

- The proof is at the level of Weil's proof of the Riemann hypothesis for finite fields and depends on Weil's exponential sum bound.
- Elementary construction/coding and intricate combinatorial/geometrical patterns.
- The Welch bound is attained.

Outline

Modeling for multi-sensor environments

- Multi-sensor environments and vector sensor and MIMO capabilities and modeling.
- Vector-valued DFTs
- Discrete time data vector $u(k)$ for a d-element array,

$$k \longmapsto u(k) = (u_0(k), \ldots, u_{d-1}(k)) \in \mathbb{C}^d.$$

We can have $\mathbb{R}^N \to GL(d, \mathbb{C})$, or even more general.

Ambiguity functions for vector-valued data

- Given $u : \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}^d$.
- For $d = 1$, $A(u) : \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}$ is

$$A(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1} u(m+k)\overline{u(k)}e^{-2\pi ikn/N}.$$

Goal

Define the following in a meaningful, computable way:

- Generalized $\mathbb{C}$-valued periodic ambiguity function $A^1(u) : \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}$
- $\mathbb{C}^d$-valued periodic ambiguity function $A^d(u)$.

The STFT is the *guide* and the *theory of frames* is the technology to obtain the goal.

Preliminary multiplication problem

- Given $u : \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}^d$.
- If $d = 1$ and $e_n = e^{2\pi in/N}$, then

$$A(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\langle u(m+k), u(k)e_{nk}\rangle.$$

Preliminary multiplication problem

To characterize sequences $\{\varphi_k\} \subseteq \mathbb{C}^d$ and compatible multiplications $*$ and $\bullet$ so that

$$A^1(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\langle u(m+k), u(k) * \varphi_{n\bullet k}\rangle \in \mathbb{C}$$

is a meaningful and well-defined *ambiguity function.* This formula is clearly motivated by the STFT.

$A^1(u)$ for DFT frames

- Given $u : \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}^d, d \leq N$.
- Let $\{\varphi_k\}_{k=0}^{N-1}$ be a DFT frame for $\mathbb{C}^d$, let $*$ be componentwise multiplication in $\mathbb{C}^d$ with a factor of $\sqrt{d}$, and let $\bullet = +$ in $\mathbb{Z}/N\mathbb{Z}$.

In this case $A^1(u)$ is well-defined by

$$\begin{aligned} A^1(u)(m,n) &= \frac{1}{N}\sum_{k=0}^{N-1}\langle u(m+k), u(k) * \varphi_{n\bullet k}\rangle \\ &= \frac{d}{N^2}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\langle \varphi_j, u(k)\rangle\langle u(m+k), \varphi_{j+nk}\rangle. \end{aligned}$$

$A^1(u)$ for cross product frames

- Take $* : \mathbb{C}^3 \times \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ to be the cross product on $\mathbb{C}^3$ and let $\{i, j, k\}$ be the standard basis.
- $i * j = k, j * i = -k, k * i = j, i * k = -j, j * k = i, k * j = -i$, $i * i = j * j = k * k = 0$. $\{0, i, j, k, -i, -j, -k, \}$ is a tight frame for $\mathbb{C}^3$ with frame constant 2. Let

$$\varphi_0 = 0, \varphi_1 = i, \varphi_2 = j, \varphi_3 = k, \varphi_4 = -i, \varphi_5 = -j, \varphi_6 = -k.$$

- The index operation corresponding to the frame multiplication is the non-abelian operation $\bullet : \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z} \longrightarrow \mathbb{Z}/7\mathbb{Z}$, where $1 \bullet 2 = 3, 2 \bullet 1 = 6, 3 \bullet 1 = 2, 1 \bullet 3 = 5, 2 \bullet 3 = 1, 3 \bullet 2 = 4$, etc.
- Thus, $u : \mathbb{Z}/7\mathbb{Z} \longrightarrow \mathbb{C}^3$ and we can write $u \times v \in \mathbb{C}^3$ as

$$u \times v = u * v = \frac{1}{2^2} \sum_{s=1}^{6} \sum_{t=1}^{6} \langle u, \varphi_s \rangle \langle v, \varphi_t \rangle \varphi_{s \bullet t}.$$

- Consequently, $A^1(u)$ is well-defined.

Generalize to quaternion groups, order 8 and beyond.

Outline

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Frame multiplication

Definition (Frame multiplication)

Let $\mathcal{H}$ be a finite dimensional Hilbert space over $\mathbb{C}$, and let $\Phi = \{\varphi_j\}_{j \in J}$ be a frame for $\mathcal{H}$. Assume $\bullet : J \times J \to J$ is a binary operation. The mapping $\bullet$ is a *frame multiplication* for Φ if there exists a bilinear product $* : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ such that

$$\forall j, k \in J, \quad \varphi_j * \varphi_k = \varphi_{j \bullet k}.$$

- The existence of frame multiplication allows one to define the ambiguity function for vector-valued data.
- There are frames with no frame multiplications.

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Harmonic frames

- Slepian (1968) - *group codes*.
- Forney (1991) - *geometrically uniform* signal space codes.
- Bölcskei and Eldar (2003) - *geometrically uniform* frames.
- Han and Larson (2000) - *frame bases and group representations*.
- Zimmermann (1999), Pfander (1999), Casazza and Kovacević (2003), Strohmer and Heath (2003), Vale and Waldron (2005), Hirn (2010), Chien and Waldron (2011) - *harmonic frames*.
- Han (2007), Vale and Waldron (2010) - *group frames, symmetry groups*.

Harmonic frames

- $(\mathcal{G}, \bullet) = \{g_1, \ldots, g_N\}$ abelian group with $\widehat{G} = \{\gamma_1, \ldots, \gamma_N\}$.
- $N \times N$ matrix with (j, k) entry $\gamma_k(g_j)$ is *character table* of $\mathcal{G}$.
- $K \subseteq \{1, \ldots, N\}, |K| = d \leq N$, and columns $k_1, \ldots, k_d$.

Definition

Given $U \in \mathcal{U}(\mathbb{C}^d)$. The *harmonic frame* $\Phi = \Phi_{\mathcal{G},K,U}$ for $\mathbb{C}^d$ is

$$\Phi = \{U\left((\gamma_{k_1}(g_j), \ldots, \gamma_{k_d}(g_j))\right) : j = 1, \ldots, N\}.$$

Given $\mathcal{G}, K$, and $U = I$. Φ is the *DFT* $-$ *FUNTF* on $\mathcal{G}$ for $\mathbb{C}^d$. Take $\mathcal{G} = \mathbb{Z}/N\mathbb{Z}$ for usual *DFT* $-$ *FUNTF* for $\mathbb{C}^d$.

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Group frames

Definition

Let $(\mathcal{G}, \bullet)$ be a finite group, and let $\mathcal{H}$ be a finite dimensional Hilbert space. A finite tight frame $\Phi = \{\varphi_g\}_{g \in \mathcal{G}}$ for $\mathcal{H}$ is a *group frame* if there exists

$$\pi : \mathcal{G} \to \mathcal{U}(\mathcal{H}),$$

a unitary representation of $\mathcal{G}$, such that

$$\forall g, h \in \mathcal{G}, \quad \pi(g)\varphi_h = \varphi_{g \bullet h}.$$

Harmonic frames are group frames.

Abelian results

Theorem (Abelian frame multiplications – 1)

Let $(\mathcal{G}, \bullet)$ be a finite abelian group, and let $\Phi = \{\varphi_g\}_{g \in \mathcal{G}}$ be a tight frame for $\mathcal{H}$. Then $\bullet$ defines a frame multiplication for Φ if and only if Φ is a group frame.

Abelian results

Theorem (Abelian frame multiplications – 2)

Let $(\mathcal{G}, \bullet)$ be a finite abelian group, and let $\Phi = \{\varphi_g\}_{g \in \mathcal{G}}$ be a tight frame for $\mathbb{C}^d$. If $\bullet$ defines a frame multiplication for Φ, then Φ is unitarily equivalent to a harmonic frame and there exists $U \in \mathcal{U}(\mathbb{C}^d)$ and $c > 0$ such that

$$cU\left(\varphi_g * \varphi_h\right) = cU\left(\varphi_g\right) cU\left(\varphi_h\right),$$

where the product on the right is vector pointwise multiplication and $*$ is defined by $(\mathcal{G}, \bullet)$, i.e., $\varphi_g * \varphi_h := \varphi_{g \bullet h}$.

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Remarks

- Given $u : \mathcal{G} \longrightarrow \mathcal{H}$, where $\mathcal{G}$ is a finite abelian group and $\mathcal{H}$ is a finite dimensional Hilbert space. The vector-valued ambiguity function $A^d(u)$ exists if frame multiplication is well-defined for a given tight frame for $\mathcal{H}$.
- There is an analogous characterization of frame multiplication for non-abelian groups (T. Andrews).
- It remains to extend the theory to infinite Hilbert spaces and groups.
- It also remains to extend the theory to the non-group case, e.g., our cross product example.

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Outline

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Frames of translates for $\mathbb{R}^d$

Let $f \in L^2(\mathbb{R}^d)$. Define

$$\Phi_f(\gamma) = \sum_{m \in \mathbb{Z}^d} |\widehat{f}(\gamma + m)|^2, \quad \gamma \in [0,1)^d$$

and

$$V_f = \overline{span}\{\tau_m f : m \in \mathbb{Z}^d\}.$$

$\Phi_f \in L^1([0,1)^d)$; and $\{\tau_m f\}$ Bessel for V_f, implies $\Phi_f \in L^2([0,1)^d)$.

Theorem 1 (J. Benedetto and Shidong Li, 1992 [8], [9] Section 3.8)

Let $f \in L^2(\mathbb{R}^d)$. Then, $\{\tau_m f : m \in \mathbb{Z}^d\}$ is a frame for V_f if and only if

$$\exists\, A, B > 0 \text{ such that } A \leq \Phi \leq B \text{ on } [0,1)^d \setminus N,$$

$N = \{\gamma \in [0,1)^d : \Phi(\gamma) = 0\}$ (N defined up to sets of measure 0).

Invariant spaces for LCGs

This is a large area with great generalization, applicability, and abstraction, and with a large number of first class contributors.*

* I have not read most of the papers and I am still friends with many of the authors. (White light contains all wavelengths of visible light. White backgrounds contain all articles and authors.)

Number theoretic LCAGs – set-up

- Let G be a LCAG with compact-open subgroup H.
- $H^{\perp}$ is compact-open: G/H is discrete; $\widehat{G}/H^{\perp}$ is discrete; $\widehat{G/H} = H^{\perp}$ and thus compact-open.
- Generally, G and $\widehat{G}$ do not have non-trivial discrete subgroups.
- *Assume* G/H and $\widehat{G}/H^{\perp}$ are countable (cleaner but stronger than necessary); and let $\mathcal{D} \subset \widehat{G}$ be a countable section of coset representatives of $\widehat{G}/H^{\perp}$.
- **Example.** Let $G = \mathbb{Q}_p$, the field of p-adic numbers, with $H = \mathbb{Z}_p$, the ring of p-adic integers. In fact, examples abound.

Translation

Our point of view is to think of translation in terms of a group of operators under composition as opposed to evaluation on an underlying discrete subgroup.

For any fixed $[x] \in G/H$, the *translation operator*,

$$\tau_{[x],\mathcal{D}} : L^2(G) \longrightarrow L^2(G),$$

is well-defined by the formula,

$$\forall\, f \in L^2(G), \quad \tau_{[x],\mathcal{D}}\, f = f * w^{\vee}_{[x],\mathcal{D}},$$

where $w_{[x],\mathcal{D}} : \widehat{G} \longrightarrow \mathbb{C},\ \gamma \mapsto \overline{(x, \lambda_\gamma)}$, and $\gamma + \lambda_\gamma = \sigma_\gamma \in \mathcal{D}$.

This translation was originally defined for our wavelet theory on local fields (2004) [1]. $w_{[x],\mathcal{D}}$ depends on $[x]$ and $\mathcal{D}$, but not on x.

$V_{\mathcal{D},f}$ and $\Phi_{\mathcal{D},f}(g)$

Take $f \in L^2(G)$ and define

$$V_{\mathcal{D},f} = \overline{\operatorname{span}}\,\{\tau_{[x],\mathcal{D}}\, f : [x] \in G/H\}$$

and

$$\forall\, g \in L^2(G), \quad \Phi_{\mathcal{D},f}(g)(\eta) = \sum_{\sigma \in \mathcal{D}} \widehat{g}(\eta+\sigma)\overline{\widehat{f}(\eta+\sigma)},\ \eta \in H^{\perp}.$$

Clearly, $\Phi_{\mathcal{D},f}(g) \in L^1(H^{\perp})$. Denote $\Phi_{\mathcal{D},f}(f)$ as $\Phi_{\mathcal{D}}(f)$.

Lemma

Let $f \in L^2(G)$, and assume the sequence, $\{\tau_{[x],\mathcal{D}}\, f : [x] \in G/H\}$, satisfies Bessel's inequality,

$$\exists\, B > 0 \text{ such that } \forall\, g \in V_f, \quad \sum_{[x] \in G/H} |\langle g, \tau_{[x],\mathcal{D}}\, f\rangle|^2 \leq B\, \|g\|_{L^2(G)}.$$

Then, $\Phi_{\mathcal{D}}(f) \in L^2(H^{\perp})$.

Frames of translates for number theoretic LCAGs

Theorem 2 (J. and R. Benedetto)

The sequence, $\{\tau_{[x],\mathcal{D}} f : [x] \in G/H\}$, is a frame for $V_{\mathcal{D},f}$ if and only if

$$\exists\, A, B > 0 \text{ such that } \quad A \leq \Phi_{\mathcal{D}}(f) \leq B \text{ on } H^{\perp} \setminus N,$$

where $N = \{\eta \in H^{\perp} : \Phi_{\mathcal{D}}(f)(\eta) = 0\}$ and N is defined up to sets of measure 0.

Idea of proof. Integrate over the compact group $H^{\perp}$ instead of the section $[0,1)^d$, sum over the section $\mathcal{D}$ instead of the discrete subgroup $\mathbb{Z}^d$, design the correct definition of translation, and pray.

Outline

Introduction

Garrett Birkhoff and John von Neumann [10] (1936) introduced *quantum logic* and the role of *lattices* to fathom "the novelty of the logical notions which quantum theory pre-supposes".

The topics for this "novelty" include:

- Heisenberg' uncertainty principle,
- Principle of non-commutativity of observations.

Their fundamental ideas led to the *Representation theorem* that, loosely speaking, allows one to treat quantum events as a lattice $L(H)$ of subspaces of a Hilbert space H over $\mathbb{R}$ or $\mathbb{C}$.

The role of Gleason's theorem and our theme/goal

Gleason's theorem [14] (1957) provides the transition from the lattice interpretation of quantum events to a validation of the *Born model* for probability in quantum mechanics.

Theme/goal:

- Define and implement *Gleason's function for orthonormal bases (ONBs) and the unit sphere* (a notion essential for his theorem) for the setting of Parseval frames and the closed unit ball.
- As a consequence, analyze and understand the *extensions* of Heisenberg's uncertainty principle in the context of a Gleason theorem for all Parseval frames, just as Gleason's original theorem was in the context of ONBs. These extensions of Heisenberg's uncertainty principle are both physically motivated and use many techniques from harmonic analysis, see, e.g., Benedetto and Heinig [4] (1992), [5] (2003).

Gleason functions

Andrew Gleason's classification of measures on closed subspaces of Hilbert spaces depends on his notion of frame functions [14] (1957). Since this is not related to the theory of frames, we shall refer to his functions as Gleason functions.

Definition (*A Gleason function for all orthonormal bases*)
Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. A *Gleason function* of weight W for the Hilbert space $\mathbb{K}^d$ is a function $f : \mathbb{K}^d \longrightarrow \mathbb{R}$ defined on the unit sphere S^{d-1} of $\mathbb{K}^d$ such that if $\{x_i\}$ is an orthonormal basis for $\mathbb{K}^d$, then

$$\sum_i f(x_i) = W.$$

Gleason functions, continued

Definition (*Frame*)
$\{x_i\}_{i=1}^N \subset \mathbb{K}^d$ is a *frame* for $\mathbb{K}^d$ if

$$\exists\, A, B > 0, \text{ such that } \forall y \in \mathbb{K}^d,\ A\,\|y\|^2 \leq \sum_{i=1}^{N} |\langle y, x_i\rangle|^2 \leq B\,\|y\|^2 .$$

If $A = B = C$, then $\{x_i\}$ is a *C-tight frame* for $\mathbb{K}^d$. If $C = 1$, then $\{x_i\}$ is a *Parseval frame* for $\mathbb{K}^d$.

Definition (*A Gleason function for Parseval frames*)
A *Gleason function* of weight W for all Parseval frames P for $\mathbb{K}^d$ is a function $f : \mathbb{K}^d \longrightarrow \mathbb{R}$ with the property that $f : B^d \longrightarrow [0, W)$, where $B^d \subset \mathbb{K}^d$ is the closed unit ball, and for which

$$\forall P = \{x_j\}_{j \in J},\ \sum_{j \in J} f(x_j) = W.$$

Gleason's theorems

A linear operator $A : \mathbb{K}^d \longrightarrow \mathbb{K}^d$ is *self-adjoint positive semi-definite operator* if

$$\forall x, y \in \mathbb{K}^d, \quad \langle A(x), y \rangle = \langle x, A(y) \rangle \geq 0.$$

Theorem 1

Let $f : S^{d-1} \longrightarrow \mathbb{R}$ be a non-negative Gleason function for all orthonormal bases for $\mathbb{K}^d$. There exists a self-adjoint positive semi-definite linear operator $A : \mathbb{K}^d \longrightarrow \mathbb{K}^d$ such that

$$\forall x \in S^{d-1}, \quad f(x) = \langle A(x), x \rangle \tag{1}$$

Remark A strong converse is more straightforward to prove: If A is a self-adjoint linear operator $A : \mathbb{K}^d \longrightarrow \mathbb{K}^d$, then the function f defined by (1) is a Gleason function for all orthonormal bases for $\mathbb{K}^d$.

Gleason's theorems, continued

A *measure on the closed subspaces of the Hilbert space* $\mathbb{K}^d$ is a function μ, that assigns, to every closed subspace of $\mathbb{K}^d$, a non-negative number such that if $\{H_i\}$ is a sequence of mutually orthogonal subspaces having closed linear span X, then

$$\mu(X) = \sum \mu(H_i).$$

Theorem 1 is used in the proof of the following, which is also true for separable infinite dimensional Hilbert spaces over $\mathbb{K}$.

Theorem 2

Let μ be a measure on the closed subspaces of $\mathbb{K}^d$, where $d \geq 3$. There exists a positive semi-definite operator $A : \mathbb{K}^d \longrightarrow \mathbb{K}^d$ such that, for all closed subspaces $H \subseteq \mathbb{K}^d$

$$\mu(H) = tr(AP_H),$$

where P_H is the orthogonal projection of $\mathbb{K}^d$ onto H.

Born model for quantum probabilities

Gleason's Theorem 2 is of interest to quantum theorists when $\mu(\mathbb{C}^d) = 1$, that is, when μ is a probability measure on the orthogonal subspaces of $\mathbb{C}^d$. In particular, Gleason's theorem reaffirms the Born interpretation for quantum probabilities, where observables are associated with the eigenvalues of a self-adjoint linear operator and probabilities of these observables are projections onto the associated eigenspaces.

Remark Gleason's Theorem 2 extends to quantum information theory, where POVMs arise naturally, also see Busch [11] (2003), cf. with role of POVMs in quantum detection by Benedetto and Kebo [6] (2008) and [3].

POVMs and Parseval frames

The following material is well known, see [6].

Definition A *positive operator valued measure* (POVM) in $\mathbb{K}^d$ is a set E of self-adjoint positive semi-definite linear operators such that $\sum_{M \in E} M = I$, where I is the identity operator for $\mathbb{K}^d$.

Proposition Let $\{x_j\}_{j \in J}$ be a Parseval frame for $\mathbb{K}^d$. Then, the set $E = \{M_j = x_j x_j^*\}_{j \in J}$ of linear operators on $\mathbb{K}^d$ is a POVM.

In fact, we know, by construction, that each M_j is self-adjoint and positive semi-definite. To show that the set resolves the identity, we need only use the Parseval condition of a tight frame. Conversely, given any POVM E we can construct a Parseval frame from the operators' respective eigenvectors. We obtain –

Proposition Let $E = \{M_j\}_{j \in J}$ be a POVM in $\mathbb{K}^d$. There exists a Parseval frame $\{x_{j_k}\}$ such that for each M_j we have $M_j = \sum_k x_{j_k} x_{j_k}^*$.

Quadratic forms and homogeneity

The spectral theorem and a straightforward calculation give:

Proposition

Let $A : \mathbb{K}^d \longrightarrow \mathbb{R}$ be a self-adjoint linear operator with trace W, and define $f : \mathbb{K}^d \longrightarrow \mathbb{R}$ as

$$\forall x \in \mathbb{K}^d, \quad f(x) = \langle A(x), x \rangle.$$

The restriction of f to B^d is a Gleason function of weight W for all Parseval frames.

Since quadratic forms as above are homogeneous functions of degree 2, we have proved:

Theorem

Let f be a Gleason function for all Parseval frames for $\mathbb{K}^d$. Then,

$$\forall x \in B^d \text{ and } \forall \alpha \in [0, 1], \quad f(\alpha x) = \alpha^2 f(x).$$

A characterization of Gleason functions for all Parseval frames

By combining Gleason's theorem with Naimark's theorem [16] (1940) (see Chandler Davis [13] (1977), Han and Larson [15] (2000), and Czaja [12] (2008) on Naimark), we can prove the following:

Theorem

Given $\mathbb{K}^d$, where $d \geq 2$, and a function, $f : \mathbb{K}^d \longrightarrow \mathbb{R}$. Then, f is a Gleason function for all Parseval frames for $\mathbb{K}^d$ if and only if there exists a self-adjoint linear operator, $A : \mathbb{K}^d \longrightarrow \mathbb{K}^d$, with positive trace W, such that

$$\forall x \in B^d, \quad f(x) = \langle A(x), x \rangle$$

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Epilogue and problems I

There are natural problems and relationships to be resolved and understood, and that we are pursuing. We list a few.

- Because of the role of the uncertainty principle in quantum mechanics and the technical role of graph theory in Schrödinger eigenmap methods for non-linear dimension reduction techniques, we are analyzing graph theoretic uncertainty principles [7], also see Paul Koprowski's thesis (2015).
- Suppose f is a Gleason function of weight W_N for all unit norm frames with N-elements for a given d-dimensional Hilbert space. Then f is constant on S^{d-1}.

  However, we have formulated the definition of a Gleason function to consider the class of all *equiangular Parseval* frames, thereby interleaving the power of Gleason's theorem with fundamental problems of equiangularity as they relate to the Welch bound and optimal ambiguity function behavior.

Epilogue and problems II

This is inextricably related to the construction of constant amplitude finite sequences with 0-autocorrelation, whose narrow-band ambiguity function is comparable to the Welch bound, e.g., see [2].

- The theory for separable infinite dimensional Hilbert spaces must be completed.
- Let $\mathcal{P}_N$ be the category of Parseval frames for $\mathbb{K}^d$, where each $P \in \mathcal{P}_N$ has $N \geq d$ elements.

  We say that $f : B^d \longrightarrow \mathbb{K}$, $B^d \subset \mathbb{K}^d$, is a *Gleason function of degree N* if

$$\exists\, W_{f,N} \in \mathbb{K} \text{ such that } \forall P = \{x_j\}_{j=1}^N \in \mathcal{P}_N, \ \sum_{j=1}^N f(x_j) = W_{f,N}.$$

$\mathcal{G}_N$ designates the category of Gleason functions of degree N.

Epilogue and problems III

There are *many* intricate geometric problems associated with $\mathcal{P}_N$ and $\mathcal{G}_N$, whose resolution we think provides further insight into generalizations of Gleason's theorem. For example, although it is clear that

$$\forall N \geq d, \quad \mathcal{G}_{N+1} \subseteq \mathcal{G}_N \subseteq \cdots \subseteq \mathcal{G}_d,$$

it is surprisingly difficult to resolve if the inclusions are proper.

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Happy Birthday, URSULA!