Measures with Locally Finite Support and Spectrum

Chenzhi Zhao

Norbert Wiener Center
Department of Mathematics
University of Maryland, College Park

Outline

Outline

Motivation: Poisson Summation Formula

- An important problem in harmonic analysis is to generalize the classical Poisson summation formula.

Theorem (Poisson summation formula)

Let Γ be a lattice in $\mathbb{R}^d$, let $\Gamma^* = \{y \in \mathbb{R}^d |\ y \cdot x \in \mathbb{Z} \textit{ for all } x \in \Gamma\}$ be the dual lattice, then for every f in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$

$$vol(\Gamma) \sum_{t\in\Gamma} f(t) = \sum_{\gamma\in\Gamma^*} \hat{f}(\gamma)$$

or equivalently

$$\left(vol(\Gamma) \sum_{x\in\Gamma} \delta_x\right)^{\wedge} = \sum_{y\in\Gamma^*} \delta_y$$

Generalization by translation and modulation

- We can apply translation and modulation to the Dirac comb $\sum_{x\in\Gamma}\delta_x$ and then take a finite sum to get what's called a generalized lattice Dirac comb:

Definition (Generalized Lattice Dirac Comb)

A *generalized lattice Driac comb* (GLDC) is a sum $\mu = \mu_1 + \dots + \mu_N$ where (a) $\mu_j = g_j\sigma_j$, $1 \le j \le N$ (b) $\sigma_j = \sum_{\gamma\in x_j+\Gamma_j}\delta_\gamma$ is a lattice Dirac comb supported by a coset $x_j + \Gamma_j$ of a lattice $\Gamma_j \subset \mathbb{R}^d$
(c) $g_j(x) = \sum_{k\in F_j} c(j,k)exp(2\pi i\omega_{j,k}x)$ is a finite trigonometric sum.

- By applying the PSF finitely many times, we see that the Fourier transform a GLDC is still a GLDC.
- Another important observation is that if Λ is the support of a GLDC, then $\Lambda_{\mathbb{Q}} = span_{\mathbb{Q}}\Lambda$ has finite dimension as a $\mathbb{Q}$ vector subspace of $\mathbb{R}^d$.

Crystalline Measure

To further generalize GLDC, Meyer proposed the following definition:

Definition (Crystalline Measure) [Meyer (2016)]

A purely atomic signed measure μ on $\mathbb{R}^d$ is a crystalline measure if
(a) the support Λ of μ is a locally finite set (the intersection of Λ with any compact set is finite)
(b) μ is a tempered distribution
(c) the distributional Fourier transform $\widehat{\mu}$ of μ is also a discrete measure supported by a locally finite set S (the spectrum of μ)

To generalize the Poisson summation formula beyond translations and modulations, our goal is to find crystalline measures that are not GLDCs. Examples have been given by Guinand, Meyer, Kolountzakis, Lev and Olevskii.

Outline

Some Number Theory Facts

- By Legendre's theorem, an integer $n \geq 0$ can be written as a sum of three squares (0^2 being admitted) if and only if n is not of the form $4^j(8k+7)$, $j, k \in \mathbb{N}$.
- Let $r_3(n)$ be the number of decompositions of the integer $n \geq 0$ into a sum of three squares, e.g., $r_3(0) = 1$, $r_3(1) = 6$, $r_3(2) = 12$, $r_3(7) = 0$, $r_3(4n) = r_3(n)$...
- E. Laudau [8] (pp.200-218), proved that

$$\sum_{0 \leq n \leq x} r_3(n) = \frac{4}{3}\pi x^{3/2} + \mathcal{O}(x^{3/4+\epsilon}), \quad x \to \infty$$

Guinand's Distribution

Theorem 1 [Guinand (1959)]

The Fourier transform of the one dimensional odd distribution

$$\sigma = -2\frac{d}{dt}\delta_0 + \sum_{n=1}^{\infty} r_3(n)n^{-1/2}(\delta_{\sqrt{n}} - \delta_{-\sqrt{n}})$$

is $\hat{\sigma} = -i\sigma$

Proof: For $x > 0$, raise the following PSF

$$\sum_{k\in\mathbb{Z}} exp(-\pi k^2 x) = x^{-1/2}\sum_{k\in\mathbb{Z}} exp(-\pi k^2/x) \tag{1}$$

to cubic power yields

$$1 + \sum_{n=1}^{\infty} r_3(n)exp(-\pi nx) = x^{-3/2} + x^{-3/2}\sum_{n=1}^{\infty} r_3(n)exp(-\pi n/x) \tag{2}$$

Proof of Theorem 1

Let $f_x(t) = texp(-\pi x t^2), t \in \mathbb{R}, x > 0$. Then f_x is odd and its Fourier transform is $\hat{f}_x(\gamma) = -ix^{-3/2}\gamma exp(-\pi\gamma^2/x)$, then (2) can be written as

$$< \sigma, f_x >= i < \sigma, \hat{f}_x > \tag{3}$$

The collection $\{f_x(t) = texp(-\pi x t^2), x > 0\}$ is total (linear span being dense) in the subspace of odd Schwartz class. Thus (3) implies $< \sigma, \phi >= i < \hat{\sigma}, \phi >$ holds for every odd Schwartz function ϕ. However, since σ itself is odd, thus $< \sigma, \phi >= i < \hat{\sigma}, \phi >$ is automatically true for even Schwartz functions ϕ. Thus, (5) is true for all Schwartz functions, since every Schwartz function can be written as a sum of an odd one and an even one. Thus we have $\hat{\sigma} = -i\sigma$.

Guinand's Measure

- Guinand's distribution σ is not a measure, a simple modification of σ yields a crystalline measure. Let $\alpha \in (0, 1)$, and set

$$\tau_\alpha = \left(\alpha^2 + \frac{1}{\alpha}\right)\sigma(t) - \alpha\sigma(\alpha t) - \sigma(t/\alpha) \tag{4}$$

- The derivative of the Dirac mass at 0 disappears from this linear combination. On the Fourier transform side

$$\hat{\tau}_\alpha(\gamma) = \left(\alpha^2 + \frac{1}{\alpha}\right)\hat{\sigma}(\gamma) - \hat{\sigma}(\gamma/\alpha) - \alpha\hat{\sigma}(\alpha\gamma) = -i\tau_\alpha \tag{5}$$

Guinand's Measure

- In particular, fix $\alpha = 1/2$ in the above construction and let $\tau = \tau_{1/2}$. A simple calculation shows that

Theorem 2 [Meyer (2016)]

The Fourier transform of the measure

$$\tau = \sum_{n=1}^{\infty} \chi(n) r_3(n) n^{-1/2} (\delta_{\sqrt{n}/2} - \delta_{-\sqrt{n}/2})$$

is $-i\tau$, where $\chi(n) = -1/2$, if $n \in \mathbb{N} \backslash 4\mathbb{N}$, $\chi(n) = 4$, if $n \in 4\mathbb{N} \backslash 16\mathbb{N}$, and $\chi(n) = 0$, if $n \in 16\mathbb{N}$

Outline

Kolountzakis's Construction

- Inspired by Meyer's work, Kolountzakis gave an example of a measure whose support and spectrum are both not contained in any finite union of arithmetic progressions.
- A measure μ is said to be translation bounded if the set $\{|\mu|(K + x) : x \in \mathbb{R}^d\}$ is bounded for each compact set $K \subset \mathbb{R}^d$.
- A translation bounded measure will always be a tempered distribution.

Theorem 3 [Kolountzakis (2015)]

There is a translation bounded measure ν of the form $\nu = \sum_{\lambda\in\Lambda} c_\lambda \delta_\lambda$, ($c_\lambda \neq 0$) such that $\Lambda \subset \mathbb{R}$ is a locally finite set and such that $\hat{\nu}$ is also a translation bounded measure of the form $\hat{\nu} = \sum_{s\in S} d_s \delta_s$, ($d_s \neq 0$) where S is also a locally finite set and such that both Λ and S are not contained in finite unions of arithmetic progressions. Therefore this Fourier pair cannot be derived by finitely many applications of the PSF.

Kolountzakis's Construction

Lemma 2

There is a function $f\colon \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, not identically zero, such that both the function and its Fourier transform $\hat{f}\colon \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ vanish in the interval

$$I = \left\{x \in \mathbb{Z}/N\mathbb{Z} : |x| \leq \tfrac{N}{10}\right\}$$

Proof: We search for $f\colon \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ which is 0 on I such that $\hat{f}$ also vanishes on I. This is a homogeneous linear system (the unknowns are the values of f off I) with more unknowns ($\sim 4N/5$ of them) then equations ($\sim N/5$ of them) so there is a non-zero solution. $\square$

Kolountzakis's Construction

Lemma 3

Suppose $M > 1$ is an integer. Then there is a non-zero measure μ of the form $\mu = \sum_{n\in\mathbb{Z}} a_n\delta_{An}$, whose Fourier transform is a measure $\hat{\mu}$ of the form $\hat{\mu} = \sum_{n\in\mathbb{Z}} b_n\delta_{Bn}$, where A, B are positive real numbers, and such that both μ and $\hat{\mu}$ are 0 in the interval $(-M, M)$. Furthermore the measures μ and $\hat{\mu}$ can be taken to be periodic and the numbers A and B may be chosen to be rational.

Proof: Let us start with the function $f\colon \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ of Lemma 2, with $N = 100M^2$

Proof of Lemma 3

Define first the measure $\tau = \sum_{n\in\mathbb{Z}} f(nmodN)\delta_n$. Then the measure τ is N-periodic whose Fourier transform given by $\hat{\tau} = \sum_{n\in\mathbb{Z}} \hat{f}(nmodN)\delta_{n/N}$, where $\hat{f}(n) = \frac{1}{N}\sum_{k=0}^{N-1} f(k)e^{-2\pi ink/N}$ is the discrete Fourier transform. To see this, write

$$\tau = \sum_{n\in\mathbb{Z}} f(nmodN)\delta_n = \sum_{k=0}^{N-1}\sum_{n\in\mathbb{Z}} f(k)\delta_{nN+k}$$

Thus by PSF, we have

$$\hat{\tau} = \sum_{k=0}^{N-1} f(k)\Big(\sum_{n\in\mathbb{Z}}\delta_{nN+k}\Big)\hat{} = \sum_{k=0}^{N-1} f(k)\frac{1}{N}e^{-2\pi ink/N}\sum_{n\in\mathbb{Z}}\delta_{\frac{n}{N}} = \sum_{n\in\mathbb{Z}} \hat{f}(nmodN)\delta_{\frac{n}{N}}$$

It follows that τ vanishes in the interval $(-\frac{N}{10}, \frac{N}{10})$ and $\hat{\tau}$ vanishes in the interval $(-\frac{1}{10}, \frac{1}{10})$.

Proof of Lemma 3

Define the measure μ to be the dilate of τ by $1/\sqrt{N}$

$$\mu = \sum_{n\in\mathbb{Z}} f(n mod N)\delta_{n/\sqrt{N}}$$

It follows that

$$\hat{\mu} = \sqrt{N} \sum_{n\in\mathbb{Z}} \hat{f}(n mod N)\delta_{n/\sqrt{N}}$$

Therefore, both μ and $\hat{\mu}$ vanish in the interval $(-\frac{\sqrt{N}}{10}, \frac{\sqrt{N}}{10}) = (-M, M)$. $\square$

Proof of Theorem 3

Now we may begin to proof Theorem 3. Take a sequence $M_n \to \infty$ and apply repeatedly Lemma 3 to obtain a sequence of periodic measures μ_n of discrete support, having also $\widehat{\mu_n}$ periodic and of discrete support and such that both μ_n and $\widehat{\mu_n}$ vanish in the interval $(-M_n, M_n)$.
Denote by T_r the translation by r and by M_a the modulation operator by a. Let $\epsilon_n \to 0$ be a $\mathbb{Q}$-linearly independent sequence. Each measure μ_n or $\widehat{\mu_n}$ has bounded total variation in any interval of unit length (since they are periodic), say by V_n. Define $D_n = V_n n^2$

Proof of Theorem 3

Consider the measure

$$\nu = \sum_{n \geq 1} \frac{1}{D_n} M_{\epsilon_n} T_{\epsilon_n} \mu_n$$

whose Fourier transform is the measure

$$\hat{\nu} = \sum_{n \geq 1} \frac{1}{D_n} T_{\epsilon_n} M_{-\epsilon_n} \widehat{\mu_n}$$

It follows that ν and $\hat{\nu}$ have bounded total variation in any interval of unit length.

Proof of Theorem 3

We need to show that the support of both μ and $\hat{\mu}$ are locally finite. Let $J = (a, b)$ be any interval. Then there is an index n_0 such that for $n \geq n_0$ we have $(a, b) \subset (-M_n + 1, M_n - 1)$, therefore the support of ν or $\hat{\nu}$ in J comes only from the contributions of the measures μ_1, μ_2,..., μ_{n_0} or $\widehat{\mu_1}$, $\widehat{\mu_2}$,..., $\widehat{\mu_{n_0}}$ and consists therefore of a finite number of points. Hence both *supp*ν and *supp*$\hat{\nu}$ are locally finite. The fact that both *supp*ν and *supp*$\hat{\nu}$ are infinite dimensional over $\mathbb{Q}$ follows from our choice of the numbers ϵ_n. $\square$

Meyer's Construction

Theorem 4 [Meyer (2016)]

There exists an odd crystalline measure μ on $\mathbb{R}$ such that its support Λ and its spectrum S have the following properties:
(i) Each finite subset of Λ_+ is linearly independent over $\mathbb{Q}$
(ii)Each finite subset of S_+ is linearly independent over $\mathbb{Q}$.
The construction is to set

$$\sigma_{(\alpha,\beta)} = \sum_{k\in\mathbb{Z}^3} \frac{exp(2\pi ik\beta)}{|k+\alpha|}\left(\delta_{|k+\alpha|} - \delta_{-|k+\alpha|}\right)$$

where $\alpha, \beta \in \mathbb{R}^3 \backslash \mathbb{Z}^3$. Then the Fourier transform of $\sigma_{(\alpha,\beta)}$ is

$$\mathcal{F}\left(\sigma_{(\alpha,\beta)}\right) = -iexp(-2\pi i\alpha\beta)\sigma_{(\beta,\alpha)}$$

Outline

Model sets

- Let $\Gamma \subset \mathbb{R}^n \times \mathbb{R}^m$ be an oblique lattice. This means that the two projections $p_1 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $p_2 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, once restricted to Γ, are injective with dense images.

Definition (Model sets)

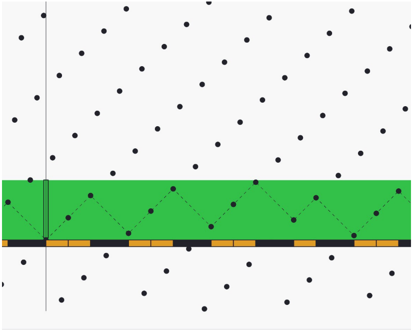
Let $K \subset \mathbb{R}^m$ be a compact set with positve measure. Then the model set Λ defined by Γ and K is

$$\Lambda = \{\lambda = p_1(\gamma) : \gamma \in \Gamma, p_2(\gamma) \in K\}$$

- Quasicrystals were discovered by Dan Shechtman in 1982. Today we know that model sets defined by cut and projection scheme can be used to model quasicrystals.

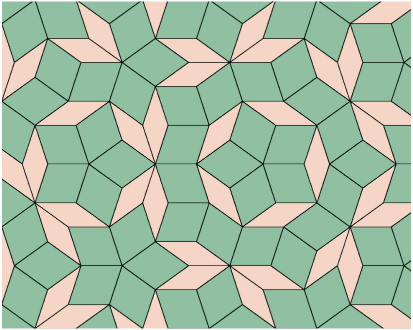
An example of a 1d quasicrystal

Figure: The Fibonacci sequence is a 1d model set



An example of a 2d quasicrystal

Figure: The Penrose tiling is a 2d model set



Some open problems

- If Λ is a model set, it can be shown that $\sum_{\lambda \in \Lambda} \delta_\lambda$ is never a crystalline measure. An open question is: does there exist a crystalline measure $\sum_{\lambda \in \Lambda} c(\lambda)\delta_\lambda$ carried by a model set Λ? What are the connections between crystalline measures and quasicrystals.
- Does there exist positive crystalline measures other than Dirac combs?
- Are crystalline measures almost periodic patterns?

Outline

[1] Andrew. P. Guinand
Concordance and the harmonic analysis of sequences.
Acta Mathematica. 101.3-4 (1959): 235-271.

[2] Yves. F. Meyer
Measures with locally finite support and spectrum.
Proceedings of the National Academy of Sciences. 113.12 (2016): 3152-3158.

[3] Yves. F. Meyer
Algebraic numbers and harmonic analysis
Elsevier. Vol. 2, 1972.

[4] Yves. F. Meyer
Guinand's measures are almost periodic distributions
Bulletin of the Hellenic Mathematical Society Volume 61, 2017: 11-20.

[5] Mihail. N. Kolountzakis

Fourier pairs of discrete support with little structure.
Journal of Fourier Analysis and Applications. 22.1 (2016): 1-5.

[6] N. Lev, and A. Olevskii
Quasicrystals with discrete support and spectrum
arXiv preprint. arXiv:1501.00085 (2014).

[7] N. Lev, and A. Olevskii
Quasicrystals and Poisson's summation formula.
Inventiones mathematicae. 200.2 (2015): 585-606.

[8] E. Landau
Collected Works, vol.6
Thales-Verlag, Essen, 1986.