Optimal coherence from finite group actions

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Outline

1. Motivation
2. Tight frames from group actions
3. ETFs with Heisenberg symmetry

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2. Tight frames from group actions
3. ETFs with Heisenberg symmetry

Motivation: Compressed sensing

Source: `mylittledrummerboys.blogspot.com`

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Underdetermined system:

$$\begin{bmatrix} \quad & \quad & \Phi & \quad & \quad \end{bmatrix} \begin{bmatrix} \\ x \\ \\ \end{bmatrix} = \begin{bmatrix} y \end{bmatrix}$$

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But! x has a lot of zero entries

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But! x has a lot of zero entries

Try a linear program:

$$\text{Minimize } \|x\|_1 \text{ subject to } \Phi x = y$$

Motivation: Compressed sensing

Rescale so the columns of Φ have unit ℓ^2-norm:

$$\Phi = \begin{bmatrix} | & | & \cdots & | \\ \varphi_1 & \varphi_2 & \cdots & \varphi_N \\ | & | & \cdots & | \end{bmatrix}$$

The **coherence** is

$$\mu(\Phi) := \max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle|.$$

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Theorem (Donoho, Elad '03)

If $\Phi x = y$ and x has no more than

$$\frac{1}{2}\left(1 + \frac{1}{\mu(\Phi)}\right)$$

nonzero entries, then x is the unique solution to the linear program

$$\textit{Minimize } \|x\|_1 \textit{ subject to } \Phi x = y.$$

Goal

Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. Given a dimension M and a number $N \geqslant M$, find a collection of unit norm vectors $\Phi = \{\varphi_i\}_{i=1}^N$ in $\mathbb{F}^M$ with minimal possible coherence.

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A collection of unit vectors $\Phi = \{\varphi_i\}_{i=1}^N$ has coherence equal to the Welch bound if and only if Φ is an equiangular tight frame.

Outline

Equiangular tight frames (ETFs)

Definition

Let $M \leqslant N$ be positive integers, and let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. The **synthesis operator** of a sequence of vectors $\{\varphi_i\}_{i=1}^N$ in $\mathbb{F}^M$ is the $M \times N$ matrix Φ with the φ_i's as its columns:

$$\Phi = \begin{bmatrix} | & | & \cdots & | \\ \varphi_1 & \varphi_2 & \cdots & \varphi_N \\ | & | & \cdots & | \end{bmatrix}$$

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- Inner products of distinct columns of Φ have constant modulus

$$|\langle \varphi_i, \varphi_j \rangle| = \begin{cases} 1 & i = j \\ \mu & i \neq j \end{cases}$$

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$$|\langle \varphi_i, \varphi_j \rangle| = \begin{cases} 1 & i = j \\ \mu & i \neq j \end{cases}$$

- The rows of Φ are orthogonal and equal norms: $\Phi\Phi^* = A\mathbf{I}$

Equiangular tight frames (ETFs)

Some applications:

- Compressed sensing
- Quantum information theory
- Wireless communication
- Phase retrieval
- Algebraic coding theory
- Signal processing
- Digital fingerprinting

The Gram matrix

Let Φ be as before, then the Gram matrix is

$$\Phi^*\Phi = \begin{bmatrix} \langle \varphi_1, \varphi_1 \rangle & \langle \varphi_2, \varphi_1 \rangle & \cdots & \langle \varphi_N, \varphi_1 \rangle \\ \langle \varphi_1, \varphi_2 \rangle & \langle \varphi_2, \varphi_2 \rangle & \cdots & \langle \varphi_N, \varphi_2 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \varphi_1, \varphi_N \rangle & \langle \varphi_2, \varphi_N \rangle & \cdots & \langle \varphi_N, \varphi_N \rangle \end{bmatrix}$$

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The collection $\{\varphi_i\}_{i=1}^N$ is an ETF if and only if

1. $\Phi\Phi^*$ is a multiple of the identity.
2. $\Phi^*\Phi$ has constant diagonal and constant modulus off-diagonal

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The collection $\{\varphi_i\}_{i=1}^N$ is an ETF if and only if

1. $\Phi^*\Phi$ is a multiple of a projection.
2. $\Phi^*\Phi$ has constant diagonal and constant modulus off-diagonal

Example: Harmonic ETF

Start with the DFT over $\mathbb{Z}_7$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega & \omega^3 & \omega^5 \\ 1 & \omega^3 & \omega^6 & \omega^2 & \omega^5 & \omega & \omega^4 \\ 1 & \omega^4 & \omega & \omega^5 & \omega^2 & \omega^6 & \omega^3 \\ 1 & \omega^5 & \omega^3 & \omega & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix}$$

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Start with the DFT over $\mathbb{Z}_7$.

$$\mathbb{Z}_7 \left\{ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \right. \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega & \omega^3 & \omega^5 \\ 1 & \omega^3 & \omega^6 & \omega^2 & \omega^5 & \omega & \omega^4 \\ 1 & \omega^4 & \omega & \omega^5 & \omega^2 & \omega^6 & \omega^3 \\ 1 & \omega^5 & \omega^3 & \omega & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix}$$

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Choose some special rows.

$$
\mathbb{Z}_7 \left\{ \begin{array}{c} 0 \\ \boxed{1} \\ \boxed{2} \\ 3 \\ \boxed{4} \\ 5 \\ 6 \end{array} \right.
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 \\
1 & \omega^2 & \omega^4 & \omega^6 & \omega & \omega^3 & \omega^5 \\
1 & \omega^3 & \omega^6 & \omega^2 & \omega^5 & \omega & \omega^4 \\
1 & \omega^4 & \omega & \omega^5 & \omega^2 & \omega^6 & \omega^3 \\
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\end{bmatrix}
$$

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Φ is an ETF, as we can tell from its Gram matrix

$$\Phi^*\Phi = \begin{bmatrix} 3 & x & x & y & x & y & y \\ y & 3 & x & x & y & x & y \\ y & y & 3 & x & x & y & x \\ x & y & y & 3 & x & x & y \\ y & x & y & y & 3 & x & x \\ x & y & x & y & y & 3 & x \\ x & x & y & x & y & y & 3 \end{bmatrix}$$

$$\begin{aligned} x &= \omega + \omega^2 + \omega^4 \\ y &= \omega^3 + \omega^5 + \omega^6 \\ &= \overline{x} \end{aligned}$$

$$x = \omega + \omega^2 + \omega^4 \qquad y = \overline{x} = \omega^3 + \omega^5 + \omega^6$$

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$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

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$$A_0 = \begin{bmatrix} 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{bmatrix}, A_1 = \begin{bmatrix} 0&1&1&0&1&0&0 \\ 0&0&1&1&0&1&0 \\ 0&0&0&1&1&0&1 \\ 1&0&0&0&1&1&0 \\ 0&1&0&0&0&1&1 \\ 1&0&1&0&0&0&1 \\ 1&1&0&1&0&0&0 \end{bmatrix}, A_2 = \begin{bmatrix} 0&0&0&1&0&1&1 \\ 1&0&0&0&1&0&1 \\ 1&1&0&0&0&1&0 \\ 0&1&1&0&0&0&1 \\ 1&0&1&1&0&0&0 \\ 0&1&0&1&1&0&0 \\ 0&0&1&0&1&1&0 \end{bmatrix}$$

$$A_1 A_2 = A_2 A_1 = 3A_0 + A_1 + A_2$$

$$A_1^T = A_2$$

Association schemes

Definition

A set of $N \times N$ matrices $\mathfrak{X} = \{A_0, \ldots, A_d\}$ with entries in $\{0, 1\}$ is called an **association scheme** if the following three conditions hold:

- $A_0 = I$
- $A_0 + \cdots + A_d = J$, where J is the all 1's matrix
- $\mathscr{A} = \operatorname{span} \mathfrak{X}$ forms a $*$-algebra under matrix multiplication.

It's a *commutative* scheme if $\mathscr{A}$ is a commutative algebra.

$$A_0 = \begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix}, A_1 = \begin{bmatrix} 0&1&1&0&1&0&0\\ 0&0&1&1&0&1&0\\ 0&0&0&1&1&0&1\\ 1&0&0&0&1&1&0\\ 0&1&0&0&0&1&1\\ 1&0&1&0&0&0&1\\ 1&1&0&1&0&0&0 \end{bmatrix}, A_2 = \begin{bmatrix} 0&0&0&1&0&1&1\\ 1&0&0&0&1&0&1\\ 1&1&0&0&0&1&0\\ 0&1&1&0&0&0&1\\ 1&0&1&1&0&0&0\\ 0&1&0&1&1&0&0\\ 0&0&1&0&1&1&0 \end{bmatrix}$$

$$A_1 A_2 = A_2 A_1 = 3A_0 + A_1 + A_2$$

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The spectral basis

Let $\mathfrak{X} = \{A_0, \ldots, A_d\}$ be a commutative association scheme.

Big idea

Find tight frames via their Gram matrices in span $\mathfrak{X}$.

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Find tight frames via their Gram matrices in $\operatorname{span} \mathfrak{X}$.

By the spectral theorem there is a set of mutually orthogonal projections

$$\hat{\mathfrak{X}} = \{E_0, \ldots, E_d\}$$

onto the maximal eigenspaces of $\mathfrak{X}$.

The projections in $\operatorname{span} \mathfrak{X}$ are exactly the sums of E_j's.

Example: The spectral basis for our scheme

$$A_0 = \begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix}, A_1 = \begin{bmatrix} 0&1&1&0&1&0&0\\ 0&0&1&1&0&1&0\\ 0&0&0&1&1&0&1\\ 1&0&0&0&1&1&0\\ 0&1&0&0&0&1&1\\ 1&0&1&0&0&0&1\\ 1&1&0&1&0&0&0 \end{bmatrix}, A_2 = \begin{bmatrix} 0&0&0&1&0&1&1\\ 1&0&0&0&1&0&1\\ 1&1&0&0&0&1&0\\ 0&1&1&0&0&0&1\\ 1&0&1&1&0&0&0\\ 0&1&0&1&1&0&0\\ 0&0&1&0&1&1&0 \end{bmatrix}$$

$$E_0 = \frac{1}{7}\begin{bmatrix} 1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1 \end{bmatrix}, E_1 = \frac{1}{7}\begin{bmatrix} 3&x&x&y&x&y&y\\ y&3&x&x&y&x&y\\ y&y&3&x&x&y&x\\ x&y&y&3&x&x&y\\ y&x&y&y&3&x&x\\ x&y&x&y&y&3&x\\ x&x&y&x&y&y&3 \end{bmatrix}, E_2 = \frac{1}{7}\begin{bmatrix} 3&y&y&x&y&x&x\\ x&3&y&y&x&y&x\\ x&x&3&y&y&x&y\\ y&x&x&3&y&y&x\\ x&y&x&x&3&y&y\\ y&x&y&x&x&3&y\\ y&y&x&y&x&x&3 \end{bmatrix}$$

$$\Phi^*\Phi = \boxed{3}\,A_0 + \boxed{x}\,A_1 + \boxed{y}\,A_2 = 7E_1$$

Schemes from group actions

Let $g_1 = (1,2,3,4,5,6,7)$ and $g_2 = (2,3,5)(4,7,6)$, and let $G = \langle g_1, g_2 \rangle \leqslant S_7$. It acts on $[7] \times [7]$:

$$g \cdot (m,n) = (g \cdot m, g \cdot n) \qquad \text{for } g \in G \text{ and } m,n \in [7].$$

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Light up the orbits as arrays. We get our scheme!

$$\begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix} \qquad \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1&2&3&4&5&6&7 \end{array} \\ \begin{bmatrix} 0&1&1&0&1&0&0\\ 0&0&1&1&0&1&0\\ 0&0&0&1&1&0&1\\ 1&0&0&0&1&1&0\\ 0&1&0&0&0&1&1\\ 1&0&1&0&0&0&1\\ 1&1&0&1&0&0&0 \end{bmatrix} \end{array} \qquad \begin{bmatrix} 0&0&0&1&0&1&1\\ 1&0&0&0&1&0&1\\ 1&1&0&0&0&1&0\\ 0&1&1&0&0&0&1\\ 1&0&1&1&0&0&0\\ 0&1&0&1&1&0&0\\ 0&0&1&0&1&1&0 \end{bmatrix}$$

Schurian schemes

Proposition

Let G be a finite group acting transitively on a set X. Let $A_0, \ldots, A_d$ be the $X \times X$ matrices that light up the orbits $\mathcal{O}_0, \ldots, \mathcal{O}_d$ of G on $X \times X$ with 1s. Then $\mathfrak{X} = \{A_0, \ldots, A_d\}$ is an association scheme, called a **Schurian scheme***.*

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Note: Let H be the stabilizer of a point. WLOG, $X = G/H$. We write $\mathfrak{X} = \mathfrak{X}(G, H)$.

Definition

We call (G, H) a **Gelfand pair** if $\mathfrak{X}(G, H)$ is commutative.

Spherical projections for Schurian schemes

Let G be a finite group acting transitively on a set $X = G/H$, with H the stabilizer of a point. The **permutation character** is given by

$$\chi(g) = |\{kH \in G/H : gkH = kH\}|.$$

It decomposes into irreducibles

$$\chi = m_1\chi_1 + \cdots + m_n\chi_n.$$

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Theorem

Each constituent χ_j determines a projection E_j in $\mathscr{A} = \operatorname{span}\mathfrak{X}(G,H)$ by the formula

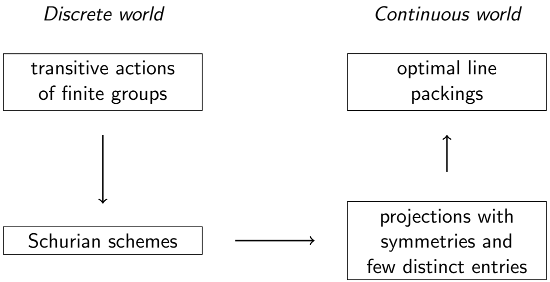
$$(E_j)_{gH,kH} = \frac{\chi_j(1)}{|G|}\sum_{h\in H}\chi_j(g^{-1}kh).$$

Any sum of E_j's is again a projection in $\mathscr{A}$. If (G,H) is a Gelfand pair, this accounts for all projections in $\mathscr{A}$.

Frames from group actions

Corollary (I., Jasper, Mixon)

Every transitive action of a finite group determines a finite number of tight frames through the spherical projections in $\mathfrak{X}(G,H)$.



- GAP code: `github.com/jwiverson/action-packings`
- Observation: Lots of these frames have optimal coherence

Homogeneous frames

Let G be a finite group, and let π be a unitary representation of G on $\mathbb{F}^M$. Suppose the orbit of a vector $\varphi \in \mathbb{F}^M$ is a frame. If

$$H = \{h \in G : \pi(h)\varphi = \varphi\},$$

then

$$\Phi = \{\pi(g)\varphi\}_{gH \in G/H}$$

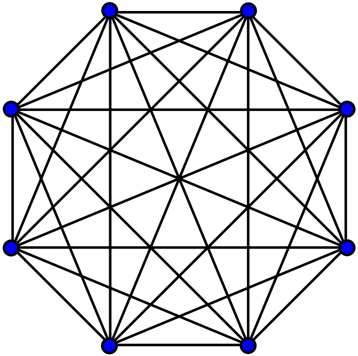
is a **homogeneous frame**, or a (G, H)-frame.

Theorem (I., Jasper, Mixon)

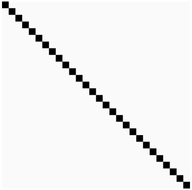
Let G be a finite group, and let $H \leqslant G$. The positive semidefinite matrices in $\operatorname{span} \mathfrak{X}(G, H)$ *are precisely the Gram matrices of (G, H)-frames.*

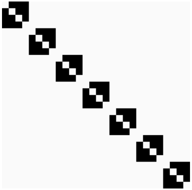
Example: Affine action on lines of $\mathbb{F}_2^3$

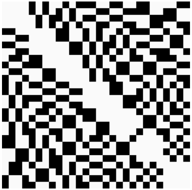
Let $G = AGL(\mathbb{F}_2^3) = GL(\mathbb{F}_2^3) \ltimes \mathbb{F}_2^3$. It acts transitively on $X = \{\text{lines in } \mathbb{F}_2^3\}$.

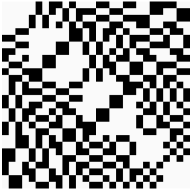


Example: Affine action on lines of $\mathbb{F}_2^3$

$A_0 =$ 

$A_1 =$ 

$A_2 =$ 

$A_3 =$ 

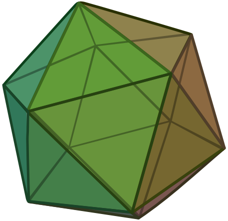
One of the spherical projections describes a 7×28 real ETF.

Example: Affine action on lines of $\mathbb{F}_2^3$

$$E = \frac{1}{12}\begin{bmatrix}
3 & - & - & - & + & - & + & - & - & + & - & + & + & + & - & - & + & + & - & - & + & + & - & - & - & - & + & + \\
- & 3 & - & - & + & - & + & - & + & - & + & - & - & - & + & + & - & - & + & + & + & + & - & - & + & + & - & - \\
- & - & 3 & - & - & + & - & + & + & - & + & - & + & + & - & - & + & + & - & - & - & - & + & + & + & + & - & - \\
- & - & - & 3 & - & + & - & + & - & + & - & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + & + \\
+ & + & - & - & 3 & - & - & - & + & + & - & - & + & - & + & - & - & + & - & + & + & + & - & - & + & - & - & + \\
- & - & + & + & - & 3 & - & - & + & + & - & - & + & - & + & - & + & - & + & - & - & - & + & + & - & + & + & - \\
+ & + & - & - & - & - & 3 & - & - & - & + & + & - & + & - & + & + & - & + & - & + & + & - & - & - & + & + & - \\
- & - & + & + & - & - & - & 3 & - & - & + & + & - & + & - & + & - & + & - & + & - & - & + & + & + & - & - & + \\
- & + & + & - & + & + & - & - & 3 & - & - & - & + & - & + & - & + & - & - & + & + & - & - & + & + & + & - & - \\
+ & - & - & + & + & + & - & - & - & 3 & - & - & + & - & + & - & - & + & + & - & - & + & + & - & - & - & + & + \\
- & + & + & - & - & - & + & + & - & - & 3 & - & - & + & - & + & - & + & + & - & - & + & + & - & + & + & - & - \\
+ & - & - & + & - & - & + & + & - & - & - & 3 & - & + & - & + & + & - & - & + & + & - & - & + & - & - & + & + \\
+ & - & + & - & + & + & - & - & + & + & - & - & 3 & - & - & - & + & + & - & - & - & + & - & + & - & + & - & + \\
+ & - & + & - & - & - & + & + & - & - & + & + & - & 3 & - & - & + & + & - & - & + & - & + & - & + & - & + & - \\
- & + & - & + & + & + & - & - & + & + & - & - & - & - & 3 & - & - & - & + & + & + & - & + & - & + & - & + & - \\
- & + & - & + & - & - & + & + & - & - & + & + & - & - & - & 3 & - & - & + & + & - & + & - & + & - & + & - & + \\
+ & - & + & - & - & + & + & - & + & - & - & + & + & + & - & - & 3 & - & - & - & + & - & - & + & - & + & + & - \\
+ & - & + & - & + & - & - & + & - & + & + & - & + & + & - & - & - & 3 & - & - & - & + & + & - & + & - & - & + \\
- & + & - & + & - & + & + & - & - & + & + & - & - & - & + & + & - & - & 3 & - & - & + & + & - & - & + & + & - \\
- & + & - & + & + & - & - & + & + & - & - & + & - & - & + & + & - & - & - & 3 & + & - & - & + & + & - & - & + \\
+ & + & - & - & + & - & + & - & + & - & - & + & - & + & + & - & + & - & - & + & 3 & - & - & - & + & - & + & - \\
+ & + & - & - & + & - & + & - & - & + & + & - & + & - & - & + & - & + & + & - & - & 3 & - & - & - & + & - & + \\
- & - & + & + & - & + & - & + & - & + & + & - & - & + & + & - & - & + & + & - & - & - & 3 & - & + & - & + & - \\
- & - & + & + & - & + & - & + & + & - & - & + & + & - & - & + & + & - & - & + & - & - & - & 3 & - & + & - & + \\
- & + & + & - & + & - & - & + & + & - & + & - & - & + & + & - & - & + & - & + & + & - & + & - & 3 & - & - & - \\
- & + & + & - & - & + & + & - & + & - & + & - & + & - & - & + & + & - & + & - & - & + & - & + & - & 3 & - & - \\
+ & - & - & + & - & + & + & - & - & + & - & + & - & + & + & - & + & - & + & - & + & - & + & - & - & - & 3 & - \\
+ & - & - & + & + & - & - & + & - & + & - & + & + & - & - & + & - & + & - & + & - & + & - & + & - & - & - & 3
\end{bmatrix}$$

Example: Projective reduction

Let $G \cong A_5$ be the symmetry group of the icosahedron, acting on the set X of vertices.



$$E = \frac{1}{4}\begin{bmatrix}
1 & -1 & x & -x & -x & x & -x & x & x & -x & -x & x \\
-1 & 1 & -x & x & x & -x & x & -x & -x & x & x & -x \\
x & -x & 1 & -1 & -x & x & x & -x & -x & x & -x & x \\
-x & x & -1 & 1 & x & -x & -x & x & x & -x & x & -x \\
-x & x & -x & x & 1 & -1 & x & -x & x & -x & -x & x \\
x & -x & x & -x & -1 & 1 & -x & x & -x & x & x & -x \\
-x & x & x & -x & x & -x & 1 & -1 & -x & x & -x & x \\
x & -x & -x & x & -x & x & -1 & 1 & x & -x & x & -x \\
x & -x & -x & x & x & -x & -x & x & 1 & -1 & -x & x \\
-x & x & x & -x & -x & x & x & -x & -1 & 1 & x & -x \\
-x & x & -x & x & -x & x & -x & x & -x & x & 1 & -1 \\
x & -x & x & -x & x & -x & x & -x & x & -x & -1 & 1
\end{bmatrix} \qquad x = \frac{1}{\sqrt{5}}$$

Source: `en.wikipedia.org/wiki/Platonic_solid`

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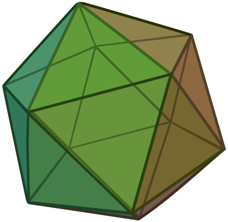

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-1 & 1 & -x & x & x & -x & x & -x & -x & x & x & -x \\
x & -x & 1 & -1 & -x & x & x & -x & -x & x & -x & x \\
-x & x & -1 & 1 & x & -x & -x & x & x & -x & x & -x \\
-x & x & -x & x & 1 & -1 & x & -x & x & -x & -x & x \\
x & -x & x & -x & -1 & 1 & -x & x & -x & x & x & -x \\
-x & x & x & -x & x & -x & 1 & -1 & -x & x & -x & x \\
x & -x & -x & x & -x & x & -1 & 1 & x & -x & x & -x \\
x & -x & -x & x & x & -x & -x & x & 1 & -1 & -x & x \\
-x & x & x & -x & -x & x & x & -x & -1 & 1 & x & -x \\
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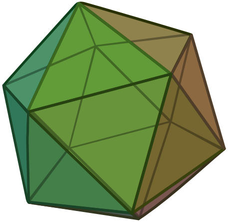

$$E = \frac{1}{4}\begin{bmatrix} 1 & x & -x & -x & x & -x \\ x & 1 & -x & x & -x & -x \\ -x & -x & 1 & x & x & -x \\ -x & x & x & 1 & -x & -x \\ x & -x & x & -x & 1 & -x \\ -x & -x & -x & -x & -x & 1 \end{bmatrix} \qquad x = \frac{1}{\sqrt{5}}$$

Now it's an ETF!

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Example: Mutually unbiased bases (MUBs)

Let $G_0 \leqslant SL(2,5)$ be the normalizer of a Sylow-2 subgroup.

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Let $G_0 \leqslant SL(2,5)$ be the normalizer of a Sylow-2 subgroup.

The affine action of $G := \mathbb{F}_5^2 \rtimes G_0$ on $\mathbb{F}_5^2$ is doubly transitive.

Hence, G acts transitively on

$$X = \{(x,y) \in \mathbb{F}_5^2 \times \mathbb{F}_5^2 : x \neq y\}.$$

Example: Mutually unbiased bases (MUBs)

Take some spherical projections and projectively reduce to get...

Three MUBs in $\mathbb{R}^4$:

$$\begin{bmatrix}
2 & 0 & 0 & 0 & - & + & + & + & - & - & - & - \\
0 & 2 & 0 & 0 & - & - & + & - & + & - & - & + \\
0 & 0 & 2 & 0 & - & - & - & + & + & + & - & - \\
0 & 0 & 0 & 2 & - & + & - & - & + & - & + & - \\
- & - & - & - & 2 & 0 & 0 & 0 & - & + & + & + \\
+ & - & - & + & 0 & 2 & 0 & 0 & - & - & + & - \\
+ & + & - & - & 0 & 0 & 2 & 0 & - & - & - & + \\
+ & - & + & - & 0 & 0 & 0 & 2 & - & + & - & - \\
- & + & + & + & - & - & - & - & 2 & 0 & 0 & 0 \\
- & - & + & - & + & - & - & + & 0 & 2 & 0 & 0 \\
- & - & - & + & + & + & - & - & 0 & 0 & 2 & 0 \\
- & + & - & - & + & - & + & - & 0 & 0 & 0 & 2
\end{bmatrix}$$

Three MUBs in $\mathbb{C}^2$:

$$\begin{bmatrix}
2 & 0 & -1+i & 1+i & -1-i & -1-i \\
0 & 2 & -1-i & 1-i & 1+i & -1-i \\
-1-i & -1+i & 2 & 0 & -1+i & 1+i \\
1-i & 1+i & 0 & 2 & -1+i & -1-i \\
-1+i & 1-i & -1-i & -1-i & 2 & 0 \\
-1+i & -1+i & 1-i & -1+i & 0 & 2
\end{bmatrix}$$

Example: A new record

The Mathieu group M_{11} acts doubly transitively on the 12 points in the Witt design W_{12}.

Take its transitive action on $X = \{(p, q) \in W_{12} : p \neq q\}$.

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Adding up some of the spherical projections (and projectively reducing), we get a tight frame of 66 vectors in $\mathbb{R}^{11}$ with coherence $\mu = 1/3$.

It beats the record in Neil Sloane's database!

Open problem

Prove this is the optimal coherence.

Example: Hoggar's lines

Let

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The Heisenberg group over $\mathbb{Z}_2^3$ is $H = \langle T^{\otimes 3}, M^{\otimes 3} \rangle$.

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Projectively reduce one of its spherical projections to get an ETF of 64 vectors in $\mathbb{C}^8$. (A SIC-POVM!)

Outline

Long-term goal: SIC-POVMs

Zauner's conjecture

For every N, the Heisenberg group over $\mathbb{Z}/N$ produces an ETF of N^2 vectors in $\mathbb{C}^N$. The ETF comes from taking the orbit of a special vector in $\mathbb{C}^N$ under the Schrödinger representation, and then projectively reducing.

- Fundamental problem in quantum information theory
- Open since 1999
- Tons of numerical evidence
- Very little theoretical progress

The Heisenberg group

Let A be a finite abelian group of odd order, with Pontryagin dual group $\hat{A}$. Let $K = A \times \hat{A}$, with additive notation.

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The **symplectic form** on K maps $[\cdot,\cdot]\colon K \times K \to C_{\exp(A)}$,

$$[(a_1, \alpha_1), (a_2, \alpha_2)] = \langle a_2, \alpha_1 \rangle \langle a_1, \alpha_2 \rangle^{-1} \qquad \left(a_i \in A,\ \alpha_i \in \hat{A}\right).$$

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The **Heisenberg group** over A is $H = K \times C_{\exp(A)}$ (as a set). Multiplication in H is given by

$$(u_1,z_1)\cdot(u_2,z_2) = (u_1+u_2, z_1 z_2 [u_1,u_2]^{1/2}) \qquad \left(u_i \in K,\ z_i \in C_{\exp(A)}\right).$$

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$$\mathsf{Sp}(K) = \{\sigma \in \mathsf{Aut}(K) : [\sigma(u), \sigma(v)] = [u, v] \text{ for all } u, v \in K\}.$$

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It acts on the Heisenberg group $H = K \times C_{\exp(A)}$ by automorphisms:

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Theorem (I., Jasper, Mixon)

$(H \rtimes \mathrm{Sp}(K), \mathrm{Sp}(K))$ *is a Gelfand pair.*

Sketch of proof

Theorem (folk)

The adjacency algebra $\mathscr{A} = \operatorname{span} \mathfrak{X}(H \rtimes K, K)$ is isomorphic to $L^2(H)^{\mathrm{Sp}(K)}$ under convolution.

New goal: Show that $L^2(H)^{\mathrm{Sp}(K)}$ is commutative.

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Lemma

The orbits of $\mathrm{Sp}(K) \leqslant \mathrm{Aut}(K)$ on K coincide with the $\mathrm{Aut}(K)$-orbits.

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Proof.

Dutta and Prasad '11: Detailed description of $\mathrm{Aut}(A)$-orbits on A.
Dutta and Prasad '15: Detailed description of $\mathrm{Sp}(K)$-orbits on K.
Observe that they coincide when we replace A with K in the first one. $\square$

Sketch of proof

Let $f_1, f_2 \in L^2(H)^{\mathrm{Sp}(K)}$ be characteristic functions of $\mathrm{Sp}(K)$-orbits in H. Then there are $\mathrm{Sp}(K)$-orbits $\mathcal{O}_1, \mathcal{O}_2 \subset K$ such that

$$f_i = \sum_{u \in \mathcal{O}_i} \delta_{(u, z_i)} \qquad (i = 1, 2).$$

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Hence,

$$\begin{aligned} f_1 * f_2 &= \sum_{u \in \mathcal{O}_1} \sum_{v \in \mathcal{O}_2} \delta_{(u, z_1) \cdot (v, z_2)} \\ &= \sum_{u \in \mathcal{O}_1} \sum_{v \in \mathcal{O}_2} \delta_{\left(u+v,\, z_1 z_2 [u,v]^{1/2}\right)}. \end{aligned}$$

Sketch of proof

For $(w, z) \in H$, we get

$$(f_1 * f_2)(w, z) = \left| \{(u, v) \in \mathcal{O}_1 \times \mathcal{O}_2 : u + v = w \text{ and } z_1 z_2 [u, v]^{1/2} = z\} \right|.$$

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By the lemma, there is some $\sigma' \in \mathrm{Sp}(K)$ with $\sigma'(w) = \sigma(w)$.
Since σ restricts to bijections on $\mathcal{O}_1$ and $\mathcal{O}_2$,

$$\begin{aligned}
&(f_1 * f_2)(w, z) = (f_1 * f_2)(\sigma'(w), z) \\
&\quad = \left|\{(u, v) \in \mathcal{O}_1 \times \mathcal{O}_2 : \sigma(u) + \sigma(v) = \sigma(w) \text{ and } z_1 z_2 [\sigma(u), \sigma(v)]^{1/2} = z\}\right| \\
&\quad = \left|\{(u, v) \in \mathcal{O}_1 \times \mathcal{O}_2 : u + v = w \text{ and } z_1 z_2 [v, u]^{1/2} = z\}\right| \\
&\quad = (f_2 * f_1)(w, z).
\end{aligned}$$

$\square$

ETFs with Heisenberg symmetry

Let $\pi\colon H \to U(L^2(A))$ be the Schrödinger representation. Define a representation ρ of H on $\mathcal{HS}(L^2(A))$ through operator multiplication

$$[\rho(u,z)](T) = \pi(u,z)\cdot T \qquad \big(T \in \mathcal{HS}(L^2(A));\ (u,z) \in H\big)$$

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Theorem (I., Jasper, Mixon)

Let P be orthogonal projection of $L^2(A)$ onto the space of even functions. After projective reduction, the orbit of P under ρ forms an ETF for its span.

Moreover, the Gram matrix of the full orbit lies in the adjacency algebra for $\mathfrak{X}(H \rtimes \mathrm{Sp}(K), \mathrm{Sp}(K))$.

Note: We get redundancy ≈ 2, not a SIC-POVM.

Thanks for your attention!

- Paper: "Optimal line packings from finite group actions"
  J.W.I., J. Jasper, D.G. Mixon
  `arXiv:1709.03558`

- GAP code: `github.com/jwiverson/action-packings`

Questions?

Source: www.sideshowtoy.com