On the recovery of measures without separation conditions

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Super-resolution

Super-resolution techniques are concerned with recovering fine details from coarse information.

There are two different categories of super-resolution:

- spectral estimation, single-snapshot, optical, diffraction,
- ~~spatial interpolation, multiple-snapshot, geometrical, image-processing~~

Applications include:

- medical imaging
- microscopy
- astronomy
- line spectral estimation
- direction of arrival estimation
- neuroscience
- geophysics

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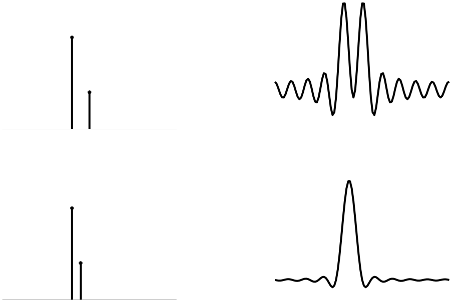
Rayleigh length

The *Rayleigh length* of an imaging system as the minimum separation between two point sources that the system can resolve.



Existing super-resolution papers

1. Point sources on $\mathbb{R}$ with continuous measurements
   - Donoho 1992
   - Demanent and Nguyen 2014
2. Well-separated point sources on $\mathbb{T}^d$ and optimization methods
   - Candès and Fernandez-Granda 2013, 2014
   - Tang, Bhaskar, Shah, and Recht 2013, 2014
   - L. 2017
3. Well-separated point sources on $\mathbb{T}^d$ and greedy methods
   - Fannjiang and Liao 2012
   - Duarte and Baraniuk 2013
4. Well-separated point sources on $\mathbb{T}$ and other methods
   - MUSIC: Liao and Fannjiang 2013
   - Matrix pencil method: Moitra 2015
5. Not well-separated point sources on $\mathbb{T}$
   - Morgenshtern and Candès 2016
   - Denoyelle, Duval and Peyré 2016
   - L. and Liao 2017
6. Curves on $\mathbb{T}^d$
   - Ongie and Jacob 2016
   - Benedetto and L. 2016

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Outline

Background

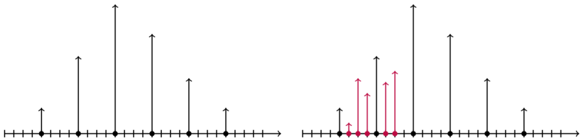
Definition (Rayleigh index)

For any discrete set $T \subseteq \mathbb{R}$, the Rayleigh index of T is the smallest number $R^*(T)$ such that any interval of length R contains at most R elements of T,

$$R^*(T) = \inf \left\{ R\colon \sup_{a \in \mathbb{R}} \#(T \cap [a, a+R)) \leq R \right\}.$$

Definition (Sparse clumps)

Let $S(N,R)$ be the set of all measures $\mu \in M(\mathbb{R})$ supported in the lattice $\{n/N\}_{n \in \mathbb{Z}}$ and has support with Rayleigh index at most R.



Background

We are given the bandwidth $\Omega > 0$ and noise level $\delta > 0$, and observe the noisy low-frequency Fourier transform of $\mu \in S(N,R)$,

$$y(\omega) = \int_{\mathbb{R}} e^{-i\omega t}\, d\mu(t) + \eta(\omega) \quad \text{for all} \quad |\omega| \le \Omega,$$

where $\|\eta\|_{L^2(-\Omega,\Omega)} \le \delta$.

Definition (Min-max error for sparse clumps)

The min-max recovery error for the sparse clumps model is

$$E(N,R,\Omega,\delta) = \inf_{\widetilde{\mu}(y,N,\Omega,R,\delta)\in S(N,R)} \sup_{\mu\in S(N,R)} \sup_{\|\eta\|_{L^2(-\Omega,\Omega)}\le\delta} \Big(\sum_{n\in\mathbb{Z}} |\widetilde{\mu}(n/N) - \mu(n/N)|^2 \Big)^{1/2}.$$

Background

Theorem (Donoho, 1992)

If N and Ω are sufficiently large, then for all R and δ, there exist $A, B > 0$ depending only on Ω, R such that

$$AN^{2R-1}\delta \leq E(N, R, \Omega, \delta) \leq BN^{2R+1}\delta.$$

- Donoho did not obtain the true dependence of $E(N, R, \Omega, \delta)$ on N, and in that same paper, he posed the problem of finding the true dependence. My opinion is that the sharp upper bound is

$$E(N, R, \Omega, \delta) \leq BN^{2R-1}\delta.$$

- The theory of super-resolution was revived about 5 years ago mainly due to a publication of Candès and Fernandez-Granda. Most recent papers focus on measures on $\mathbb{T}^d$ not $\mathbb{R}^d$.

Discrete model

Suppose there is a collection of S point sources located on a grid,

$$\mu = \sum_{n=0}^{N-1} x_n \delta_{\frac{n}{N}} \quad \text{where} \quad x \in \mathbb{C}_S^N.$$

We observe noisy low frequency Fourier coefficients,

$$y_m = \int_{\mathbb{T}} e^{-2\pi imt}\, d\mu(t) + z_m, \quad \text{for} \quad 0 \le m \le M-1,$$

where z is some unknown noise.

Important physical quantities:

- Rayleigh length $1/M$
- Grid width $1/N$
- Super-resolution factor N/M

Failure of compressed sensing

The measurements can be written as the linear system

$$y = \Phi x + z, \quad \text{where} \quad \Phi_{m,n} = e^{-2\pi i mn/N}.$$

Assuming that $\|z\| < \delta$, we could try compressed sensing techniques such as

$$\min_{\widetilde{x} \in \mathbb{C}^N} \|\widetilde{x}\|_1 \quad \text{such that} \quad \|\Phi\widetilde{x} - y\| \leq \delta.$$

When $N \gg M$, the measurement matrix $\Phi \in \mathbb{C}^{M\times N}$ fails to satisfy the conditions for standard compressed sensing theory, such as RIP and incoherence.

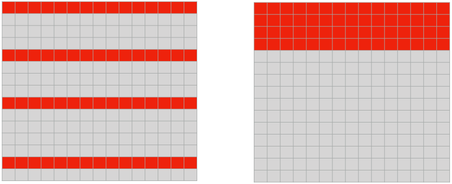


Figure : Sensing matrix for compressed sensing (left) and super-resolution (right)

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Min-max error

Definition (Min-max error)

The min-max error for the discrete model is

$$E(M,N,S,\delta) = \inf_{\substack{\widetilde{x}(y,M,N,S,\delta)\in\mathbb{C}^N \\ y=\Phi x+z}} \sup_{x\in\mathbb{C}^N_S} \sup_{\substack{z\in\mathbb{C}^M \\ \|z\|\le\delta}} \|\widetilde{x}-x\|.$$

Here, the infimum is taken over all $\widetilde{x}$ depending on the known information, M, N, S, δ and $y = \Phi x + z$, and in particular, $\widetilde{x}$ is selected independently of the unknown information x and z.

- The min-max error is method agnostic: If $x_{\text{alg}} = x_{\text{alg}}(y, M, N, S, \delta)$ is chosen according to some algorithm, then necessarily

$$\sup_{x\in\mathbb{C}^N_S} \sup_{\substack{z\in\mathbb{C}^M \\ \|z\|\le\delta}} \|x_{\text{alg}} - x\| \ge E(M,N,S,\delta).$$

- The min-max error is a strong way of measuring the performance of an algorithm because the supremum is taken over all possible S-sparse vectors and δ bounded noise.

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Sharp estimate on the min-max error

Corollary (L. and Liao, 2017)

For any integer $S > 0$, there exist $A, B > 0$ depending only on S such that for all sufficiently large integers M and N,

$$A \frac{1}{\sqrt{M}} \left(\frac{N}{M}\right)^{2S-1} \delta \leq E(M, N, S, \delta) \leq B \frac{1}{\sqrt{M}} \left(\frac{N}{M}\right)^{2S-1} \delta.$$

Implications:

- As expected, the super-resolution factor N/M governs the difficulty of recovering point sources at fine scales.
- Noise level δ needs to be small in comparison to $(N/M)^{2S-1}$ for the min-max error to be reasonably small.
- Perhaps “uniform” super-resolution recovery is hopeless? Maybe the best we can do is a theory that holds for a small subset of vectors.

Min-max error and smallest singular value

Definition (S-lower restricted isometry constant)

Let $S \leq M \leq N$. The S-lower restricted isometry constant is

$$\Theta(M,N,S) = \min_{|T|=S} \sigma_{\min}(\Phi_T),$$

where $\Phi_T \in \mathbb{C}^{M\times|T|}$ is the restriction of Φ to the columns indexed by T and $\sigma_{\min}(\Phi_T)$ is its smallest singular value.

Proposition (Demanet and Nguyen, 2015)

If $2S \leq M \leq N$ and $\delta > 0$, then

$$\frac{\delta}{2\Theta(M,N,2S)} \leq E(M,N,S,\delta) \leq \frac{2\delta}{\Theta(M,N,2S)}.$$

Sharp estimate on lower restricted isometry constant

Theorem (L. and Liao, 2017)

For any integer $S > 0$, there exist constants $A, B > 0$ depending only on S such that for all sufficiently large integers M and N,

$$A\sqrt{M}\left(\frac{M}{N}\right)^{S-1} \leq \Theta(M,N,S) \leq B\sqrt{M}\left(\frac{M}{N}\right)^{S-1}.$$

About the numerology:

- Each column of Φ has Euclidean norm $\sqrt{M}$, which explains the $\sqrt{M}$ term.
- In view of the connection between imaging, it makes sense from a physical point of view that Θ only depends on the super-resolution factor N/M.
- The singular values of Φ_T and $\Phi_{\widetilde{T}}$ are identical whenever $\widetilde{T} = T + a \mod N$ and $a \in \mathbb{Z}$, so WLOG $0 \in T$. Even though $|T| = S$ one of its columns is already fixed.

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Lower bound

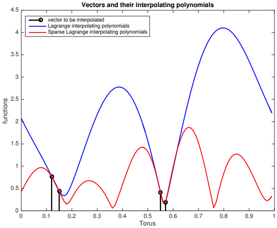
Definition (Polynomial interpolation space)

Let $S \leq M \leq N$. For each $v \in \mathbb{C}^N_S$, let $P(M,v)$ be the set of trigonometric polynomials f such that supp$(\widehat{f}) \subseteq \{0,1,\ldots,M-1\}$ and $f(n/N) = v_n$ for all $n \in$ supp(v).

Proposition: Duality (L. and Liao, 2017)

Let $S \leq M \leq N$. For any support set T of cardinality S,

$$\frac{1}{\sigma_{\min}(\Phi_T)} = \sup_{\substack{\|v\|=1 \\ \mathrm{supp}(v)\subseteq T}} \inf_{f\in P(M,v)} \|f\|_{L^2}.$$



Lower bound

Proposition: Sparse Lagrange polynomials (L. and Liao, 2017)

Fix any integer $S > 0$. There exists $C > 0$ depending only on S such that for all sufficiently large M and N and any support set T with cardinality S, there exists a family of trigonometric polynomials $\{H_n\}_{n\in T}$ such that for all $n \in T$,

- $H_n(\frac{m}{N}) = \delta_{m,n}$ for all $m \in T$,
- supp$(\widehat{H_n}) \subseteq \{0, 1, \dots, M-1\}$,
- $\|H_n\|_{L^2(\mathbb{T})} \leq C \dfrac{1}{\sqrt{M}} \left(\dfrac{N}{M}\right)^{S-1}$.

We call $\{H_n\}_{n\in T}$ the sparse Lagrange polynomials adapted to T.

The Lagrange polynomials $\{L_n\}_{n\in T}$ also satisfy the first and second properties, but it turns out that without any additional assumptions on T, the best one can do is

$$\|L_n\|_{L^2(\mathbb{T})} \leq C_S N^{S-1}.$$

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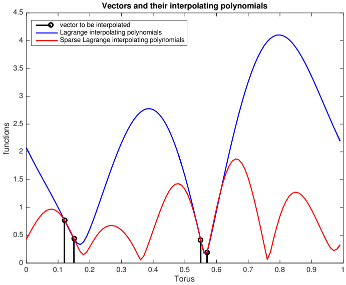
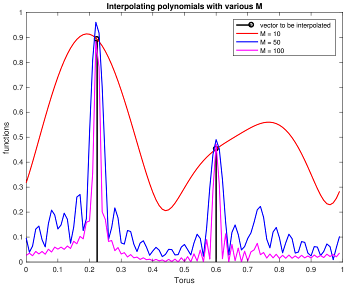
Lower bound


Vectors and their interpolating polynomials
vector to be interpolated
Lagrange interpolating polynomials
Sparse Lagrange interpolating polynomials
functions
Torus


Interpolating polynomials with various M
vector to be interpolated
M = 10
M = 50
M = 100
functions
Torus

Lower bound

Fix any support set T with cardinality S and let $\{H_n\}_{n\in T}$ be the sparse Lagrange polynomials adapted to T. For any unit norm $v \in \mathbb{C}^N$ supported in T, define the interpolating polynomial

$$H(v) = \sum_{n\in T} v_n H_n \in P(M,v).$$

By the duality principle and Cauchy-Schwarz,

$$\frac{1}{\sigma_{\min}(\Phi_T)} \leq \sup_{\substack{\|v\|=1 \\ \text{supp}(v)\subseteq T}} \|H(v)\|_{L^2(\mathbb{T})} \leq \Big(\sum_{n\in T} \|H_n\|^2_{L^2(\mathbb{T})}\Big)^{1/2} \leq C\sqrt{\frac{S}{M}}\Big(\frac{N}{M}\Big)^{S-1}.$$

This inequality holds for all support sets with cardinality S, which yields the desired lower bound for $\Theta(M,N,S)$.

Upper bound

We first write

$$\Theta(M,N,S) = \min_{|T|=S} \sigma_{\min}(\Phi_T) = \min_{|T|=S} \inf_{\substack{u\neq 0 \\ \mathrm{supp}(u)\subseteq T}} \frac{\|\Phi u\|}{\|u\|}.$$

We are looking for a $u \in \mathbb{C}_S^N$ such that $\widehat{u}$ is small on $\{0,1,\dots,M-1\}$.

If v is well-localized bump near the origin, then an appropriate modulation of v has Fourier transform that is small near the origin.

Consider the vector (Donoho 1992),

$$u_n = \begin{cases} (-1)^n \binom{S-1}{n} & \text{if } n = 0,1,\dots,S-1, \\ 0 & \text{otherwise.} \end{cases}$$

Under the assumptions of the theorem,

$$\Theta(M,N,S) \leq \frac{\|\Phi u\|}{\|u\|} \leq B\sqrt{M}\left(\frac{M}{N}\right)^{S-1}.$$

Conclusions about the discrete problem

Summary:

- Proved a sharp estimate on the lower restricted isometry constant and min-max recovery error.
- The noise needs to be extremely small in order to recover arbitrary sparse vectors.
- Constructed a new family of interpolating trigonometric polynomials.

Future work:

- Do any of the current super-resolution algorithms achieve the min-max error?
- What is the exact dependence of $\Theta(M,N,S)$ on S?
- What about arbitrary Vandermonde matrices with nodes on the circle?
- What about weaker ways of measuring the recovery rate, say probabilistic models?

Outline

Continuous model

Let $\mu \in M(\mathbb{T}^d)$. Cases of interest:

- "Off-the-grid" point sources: $\mu = \sum_{n=1}^{N} a_n \delta_{x_n}$
- "Cartoon-like" images: $\mu = \sum_{n=1}^{N} a_n \sigma_n$, where σ_n is a surface measure

Let $\Lambda \subseteq \mathbb{Z}^d$ be a finite set. Cases of interest:

- Uniform sampling: $\Lambda = \{-M, \dots, M\}^d$ for some $M > 0$
- Non-uniform sampling: no assumptions on Λ beyond finiteness

Suppose we observe

$$F(m) = \widehat{\mu}(m) = \int_{\mathbb{T}^d} e^{-2\pi i m \cdot x} \, d\mu(x), \quad \text{for all} \quad m \in \Lambda.$$

Background

Question (Exact recovery for measures)

What kinds of μ can we recover from the spectral data F on Λ?

Heuristic: If μ is “not complicated”, then we expect total variation minimization to work.

Definition (TV-min)

Given F on a finite set $\Lambda \subseteq \mathbb{Z}^d$, the total variation minimization problem is

$$\inf \|\nu\|_{\mathsf{TV}} \quad \text{such that} \quad \nu \in M(\mathbb{T}^d) \quad \text{and} \quad F(m) = \widehat{\nu}(m) \quad \text{for all} \quad m \in \Lambda.$$

Remark: The pre-dual of TV-min can be rewritten as a semi-definite program. The latter can be numerically solved in polynomial time and gives information about the support of the solutions.

Background

Theorem (Candès and Fernandez-Granda, 2014)

There exists a constant $C > 0$ depending only on the dimension d, such that all sufficiently large integers $M > 0$, the following holds. Let $\Lambda = \{-M, \dots, M\}^d$ and $\mu \in M(\mathbb{T}^d)$ be a discrete measure such that for any distinct $x, y \in \mathrm{supp}(\mu)$, we have

$$\sup_{1 \le j \le d} |x_j - y_j|_{\mathbb{T}} \ge \frac{C}{M}.$$

If $F(m) = \widehat{\mu}(m)$ for all $m \in \Lambda$, then μ is the unique solution to TV-min.

- The measure μ can be exactly recovered without any prior assumptions on its support!
- The current best result for the implicit constant for $d = 1$ is $C = 1.26$, established by Fernandez-Granda in 2016.
- In $d = 1$, the Rayleigh length for this model is $O(1/M)$ but this result only applies to discrete measures with separation $O(1/M)$...

Background

Questions

If $\#\Lambda = O(M)$, can one recover information at scales $\ll 1/M$? What happens if there is no separation assumption?

Recall the well-known dual characterization of solutions to TV-min.

Proposition: Duality

Let $\mu \in M(\mathbb{T}^d)$, $\Lambda \subseteq \mathbb{Z}^d$ be a finite set, and $F(m) = \widehat{\mu}(m)$ for all $m \in \Lambda$. Then μ is a solution to TV-min if and only if there exists a $\varphi \in C^\infty(\mathbb{T}^d)$ such that

- supp$(\widehat{\varphi}) \subseteq \Lambda$
- $\|\varphi\|_{L^\infty(\mathbb{T}^d)} \leq 1$
- $\varphi =$ sign(μ) μ-a.e.

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Beurling super-resolution

Beurling studied TV-min for $\mu \in M(\mathbb{R})$ instead of $M(\mathbb{T}^d)$. He observed that there is a "uniform support" property built into TV-min for the problem on $\mathbb{R}$. The same principle holds for $\mathbb{T}^d$.

Proposition: Uniform support (Benedetto and L., 2016)

Given F on the finite set $\Lambda \subseteq \mathbb{Z}^d$, let $\varepsilon > 0$ be the minimum value attained in TV-min. There exists a $\varphi \in C^\infty(\mathbb{T}^d)$ such that

- supp$(\widehat{\varphi}) \subseteq \Lambda$
- $\|\varphi\|_{L^\infty} \leq 1$
- $|\sum_{m\in\Lambda} \widehat{\varphi}(m)F(m)| = \varepsilon$
- all solutions to TV-min are supported in $\{x \in \mathbb{T}^d : |\varphi(x)| = 1\}$.

Beurling super-resolution

Theorem (Bendetto and L., 2016)

Given F on the finite set $\Lambda \subseteq \mathbb{Z}^d$, let $\varepsilon > 0$ be the minimum value attained in the TV-min problem and let

$$\Gamma = \{m \in \Lambda \colon |F(m)| = \varepsilon\}.$$

1. Suppose $\Gamma = \emptyset$. Then, there exists a closed set S of d-dimensional Lebesgue measure zero such that each solution to TV-min is a singular measure supported in S.
2. Suppose $\#\Gamma \geq 2$. For each distinct pair $m, n \in \Gamma$, define $\alpha_{m,n} \in \mathbb{R}/\mathbb{Z}$ by $e^{2\pi i \alpha_{m,n}} = F(m)/F(n)$. Define the closed set,

$$S = \bigcap_{\substack{m,n \in \Gamma \\ m \neq n}} \{x \in \mathbb{T}^d \colon x \cdot (m-n) + \alpha_{m,n} \in \mathbb{Z}\},$$

which is an intersection of $\binom{\#\Gamma}{2}$ periodic hyperplanes. Then, each solution to TV-min is a singular measure supported in S.

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Beurling super-resolution

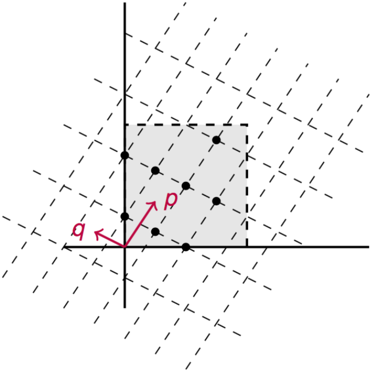

Figure : An illustration of the second statement in the theorem. The hyperplanes are represented by the dashed lines. The vectors $p = (1/4, 3/8)$ and $q = (-1/4, 1/8)$ are normal to the hyperplanes. All solutions to TV-min are supported in S, which is represented by the black dots.

Example: $\#\Gamma \geq 2$

This example shows that the second statement of the theorem is optimal, and also illustrates the importance of geometry when working in higher dimensions.

- Suppose $F(m) = \widehat{\mu}(m)$ for all $m \in \Lambda$, where $\mu = \delta_{(0,0)} + \delta_{(1/2,1/2)}$ and $\Lambda = \{-1, 0, 1\}^2 \setminus \{(1, -1), (-1, 1)\}$.
- Some calculations show that $\varepsilon = 2$ and $\Gamma = \{(-1, -1), (0, 0), (1, 1)\}$.
- According to the theorem, every solution is supported in the set,

$$S = \{x \in \mathbb{T}^2 \colon x_1 + x_2 = 1\}.$$

  Let σ_S be the surface measure of the Borel set S. We readily verify that $\sqrt{2}\sigma_S$ is also a solution.
- For any $a \in \mathbb{R}$ and any integer $N \geq 2$, the discrete measure

$$\frac{2}{N} \sum_{n=0}^{N-1} \delta_{\left(a+\frac{n}{N}, 1-a-\frac{n}{N}\right)}$$

  is also a solution.

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Example: $\#\Gamma = 1$

We cannot say anything about the case $\#\Gamma = 1$ because it is associated with pathological behaviors. It is possible that there exist uncountably many discrete and absolutely continuous solutions to TV-min.

- Suppose $F(m) = \widehat{\mu}(m)$ for $m \in \Lambda$, where $\mu = \delta_0 + \delta_{1/2}$ and $\Lambda = \{-1, 0, 1\}$.
- Some calculations show that $\varepsilon = 2$ and $\Gamma = \{0\}$.
- For any $a \in \mathbb{T}$ and any integer $N \geq 2$, the discrete measure $\frac{2}{N}\sum_{n=0}^{N-1} \delta_{a+\frac{n}{N}}$ is also a solution.
- For any integer $N \geq 2$ and $0 < a \leq (2N+2)/(3N+1)$, define the sequence $\{a_n\}_{n\in\mathbb{Z}}$, where

$$a_n = \begin{cases} 2 & \text{if } n = 0, \\ a\left(1 - \frac{|n|}{N+1}\right) & \text{if } 2 \leq |n| \leq N, \\ 0 & \text{otherwise.} \end{cases}$$

  The non-negative real-valued function

$$f(x) = 2 + 2\sum_{n=2}^{N} a_n \cos(2\pi n x)$$

  is a positive absolutely continuous solution.

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Back to minimum separation

In general, some separation assumption is necessary in order to recover a discrete measure using TV-min.

- Let $\Lambda \subseteq \mathbb{Z}$ be any finite set.
- For any $0 < a < 1/2$, let $\mu_a = \delta_0 - \delta_a$. Note that $\|\mu_a\|_{\mathrm{TV}} = 2$.
- Suppose $F(m) = \widehat{\mu_a}(m)$ for all $m \in \Lambda$.
- Let ν_a be the absolutely continuous measure,

$$\nu_a(x) = \sum_{m\in\Lambda} \widehat{\mu_a}(m) e^{2\pi i m x}.$$

By construction, $\widehat{\nu_a}(m) = \widehat{\mu_a}(m)$ for all $m \in \Lambda$. In the limit $a \to 0$,

$$\|\nu_a\|_{\mathrm{TV}} = \int_{\mathbb{T}^d} \left| \sum_{m\in\Lambda} \widehat{\mu_a}(m) e^{2\pi i m x} \right| dx \to 0.$$

By taking a sufficiently small, we have $\|\nu_a\|_{\mathrm{TV}} < \|\mu_a\|_{\mathrm{TV}}$, so μ_a is <u>not</u> a solution to TV-min.

Conclusions

Summary:

- Incorporated Beurling's ideas in the development of a new theory of super-resolution for general measures.
- Constructed specific examples to show that the theorem cannot be improved without additional assumptions on F or Λ.
- Obtained a better understanding of the capabilities of the TV-min approach.

Future work:

- What about the super-resolution of singular continuous measures using other recovery techniques, such as subspace methods?
- Is it possible to extend this approach to the noise case? What is a natural way to define the error for this setting?

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