Zak transform analysis of shift-invariant subspaces

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Outline

Group frames

Example: Dynamical sampling

Let $\mathcal{H}$ be a Hilbert space evolving discretely under a unitary V. Fix "sensors" $\{f_j\}_{j\in I}$ in $\mathcal{H}$. Measure evolutions of $g \in \mathcal{H}$:

$$Tg = \{\langle V^k g, f_j\rangle\}_{k\in\mathbb{Z}, j\in I} = \{\langle g, V^{-k} f_j\rangle\}_{k\in\mathbb{Z}, j\in I}$$

We can stably recover $g \in \mathcal{H}$ from Tg if and only if $\{V^k f_j\}_{k\in\mathbb{Z}, j\in I}$ is a frame for $\mathcal{H}$.

V defines a representation $\pi\colon \mathbb{Z} \to U(\mathcal{H})$, $\pi(k) = V^k$. We want a frame

$$\{V^k f_j\}_{k\in\mathbb{Z}, j\in I} = \{\pi(k) f_j\}_{k\in\mathbb{Z}, j\in I}$$

that is the "orbit" of the sensors $\{f_j\}_{j\in I}$.

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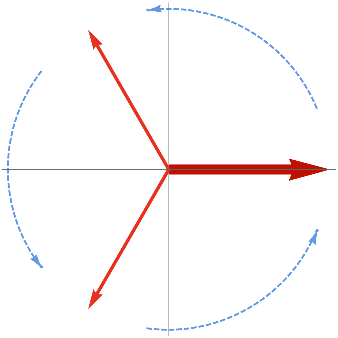
Example

Let $\mathbb{Z}/3$ act on $\mathbb{R}^2$ by rotation



Spin around $f = (1, 0)$ to get Mercedes-Benz

Group frames

Definition

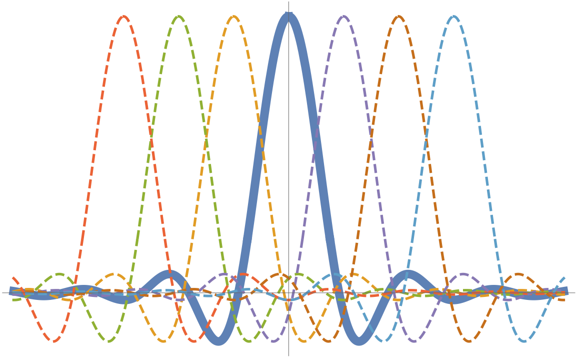
Let H be a locally compact group with left Haar measure dx. Let $\pi\colon H \to U(\mathcal{H}_\pi)$ be a unitary representation on $\mathcal{H}_\pi$. A countable family $\mathscr{A} = \{f_j\}_{j\in I} \subseteq \mathcal{H}_\pi$ *generates* a *group frame* under π if $\exists\, A, B > 0$ s.t.

$$A \|g\|^2 \leq \sum_{j\in I} \int_H |\langle g, \pi(x) f_j\rangle|^2 dx \leq B \|g\|^2 \qquad \text{for all } g \in \mathcal{H}_\pi.$$

In other words, the joint orbit of $\mathscr{A}$ is a (continuous) frame for $\mathcal{H}_\pi$.

Example

Let $\mathbb{Z}$ act on $L^2(\mathbb{R})$ by integer shifts, $[\pi(k)f](x) = f(x-k)$



The orbit of a scaling function f is $\{f(\cdot + k)\}_{k\in\mathbb{Z}}$, as in an MRA

Group frames

Great Big Question

Given $\pi \colon H \to U(\mathcal{H}_\pi)$, which families $\mathscr{A} \subseteq \mathcal{H}_\pi$ generate group frames?

At best, $\mathscr{A} = \{f_j\}_{j \in I}$ generates a frame for the invariant subspace

$$S(\mathscr{A}) := \overline{\operatorname{span}}\{\pi(x) f_j : x \in H, j \in I\}$$

Proof

If $S(\mathscr{A}) \neq \mathcal{H}_\pi$, then $\exists g \perp S(\mathscr{A})$, $g \neq 0$, and

$$\sum_{j \in I} \int_H |\langle g, \pi(x) f_j \rangle|^2 dx = 0 \ngeq A \|g\|^2 .$$

Group frames

Great Big Question 1

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Great Big Question 2

What are the invariant subspaces of $\mathcal{H}_\pi$?

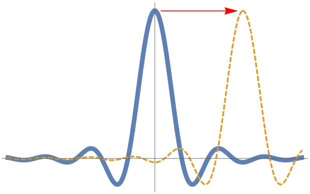
SI spaces

Fix a locally compact group $G \supseteq H$.

Left translation operator for $y \in G$:

$$L_y : L^2(G) \to L^2(G),$$

$$(L_y f)(x) = f(y^{-1}x) \qquad (x, y \in G).$$



Definition

A subspace $V \subseteq L^2(G)$ is H-shift invariant (H-SI) if

$$f \in V \implies L_y f \in V \quad \text{for all } y \in H.$$

SI spaces

Theorem (Bownik, JI – in preparation)

Let $\pi\colon H \to U(\mathcal{H}_\pi)$ be a representation, and let $\{f_j\}_{j\in I} \subseteq \mathcal{H}_\pi$ be a countable family that generates a group frame for $\mathcal{H}_\pi$. If G is any second countable group containing H as a closed subgroup of index $[G : H] \geq |I|$, then there is an isometric embedding $T\colon \mathcal{H}_\pi \to L^2(G)$ s.t.

$$T\pi(y) = L_y T \quad \text{for all } y \in H.$$

Corollary

Up to unitary equivalence, all group frames are given by shifts in shift-invariant spaces.

Proof.

T maps $\mathcal{H}_\pi$ unitarily onto the shift-invariant space $T(\mathcal{H}_\pi)$ while sending the group frame $\{\pi(y)f_j\}_{y\in G, j\in I}$ to $\{L_y Tf_j\}_{y\in H, j\in I}$. □

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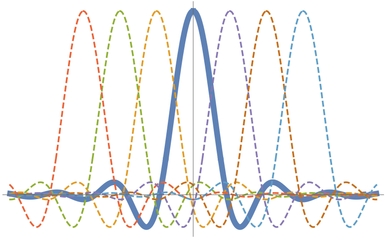
Example

We have $\mathbb{Z} \subseteq \mathbb{R}$ as a closed subgroup of infinite index. Hence:

Corollary (Bownik, JI)

Suppose $\mathcal{H}$ is a Hilbert spacing evolving under a unitary V for which there exist "sensors" $\{f_j\}_{j\in I}$ that produce a frame $\{V^k f_j\}_{j\in I}$. Then up to unitary equivalence $\mathcal{H}$ is a shift-invariant subspace of $L^2(\mathbb{R})$ and V is the shift

$$(Vg)(x) = g(x-1).$$



Great big questions, rephrased

Great Big Question 1'

Given a locally compact subgroup pair $H \subseteq G$, which families $\mathscr{A} \subseteq L^2(G)$ generate group frames under shifts by H?

Great Big Question 2'

What are the H-SI subspaces of $L^2(G)$?

This talk: Focus on G and H both abelian, second countable.

Outline

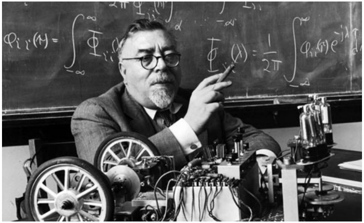
Example: $\mathbb{R}^n \subseteq \mathbb{R}^n$

Problem

Which subspaces of $L^2(\mathbb{R}^n)$ are invariant under translation?

Solution (Wiener, 1932)

Use the Fourier transform!



Key fact: Translation $\to$ Modulation

$$(L_y f)\hat{}(x) = e^{-2\pi i y \cdot x} \cdot \hat{f}(x)$$

Given $E \subseteq \mathbb{R}^n$, define $V_E = \{f \in L^2(\mathbb{R}^n) : \hat{f}(x) = 0 \text{ for a.e. } x \notin E\}$.

It's translation invariant:

$$f \in V_E,\ y \in \mathbb{R}^n \implies (L_y f)\hat{}(x) = e^{-2\pi i y \cdot x} \cdot \hat{f}(x) = 0 \text{ for } x \notin E.$$

Wiener: That's every TI space!

Example: $\mathbb{Z}^n \subseteq \mathbb{R}^n$

Fiberization operator:

$$\mathcal{T}\colon L^2(\mathbb{R}^n) \to L^2([0,1)^n; \ell^2(\mathbb{Z}^n)),$$

$$(\mathcal{T}f)(x) = \left\{ \hat{f}(x+k) \right\}_{k\in\mathbb{Z}^n} \qquad \left(f \in L^2(\mathbb{R}^n),\, x \in [0,1)^n\right).$$

Composition of unitaries $L^2(\mathbb{R}^n) \cong L^2(\mathbb{R}^n) \cong L^2([0,1)^n; \ell^2(\mathbb{Z}^n))$:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi = \int_0^1 \sum_{k\in\mathbb{Z}} |\hat{f}(\xi+k)|^2 d\xi$$

Example: $\mathbb{Z}^n \subseteq \mathbb{R}^n$

Fiberization operator:

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Integer shifts $\to$ modulation:

$$(\mathcal{T}L_m f)(x) = \left\{ (L_m f)^\wedge(x+k) \right\}_{k\in\mathbb{Z}^n} = \left\{ e^{-2\pi i m\cdot(x+k)} \cdot \hat{f}(x+k) \right\}_{k\in\mathbb{Z}^n}$$

$$= \left\{ e^{-2\pi i m\cdot x} \cdot \hat{f}(x+k) \right\}_{k\in\mathbb{Z}^n} = e^{-2\pi i m\cdot x} \cdot (\mathcal{T}f)(x).$$

Example: $\mathbb{Z}^n \subseteq \mathbb{R}^n$

Fiberization operator:

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Key properties: (1) unitary; (2) integer shifts $\to$ modulation

$$(\mathcal{T}L_k f)(x) = e^{-2\pi i k\cdot x} \cdot (\mathcal{T}f)(x) \qquad \left(f \in L^2(\mathbb{R}^n),\, k \in \mathbb{Z}^n,\, x \in [0,1)^n \right).$$

Which subspaces of $L^2([0,1)^n; \ell^2(\mathbb{Z}^n))$ are *modulation* invariant (MI)?

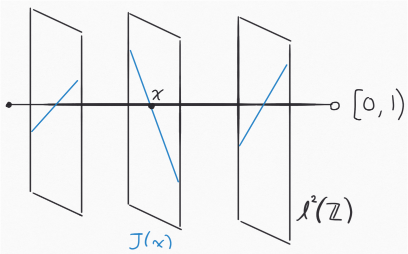
Example: $\mathbb{Z}^n \subseteq \mathbb{R}^n$

Definition

A *range function* maps

$$J\colon [0,1)^n \to \{\text{closed subspaces of } \ell^2(\mathbb{Z}^n)\}.$$

Let $P(x)$ be projection onto $J(x) \subseteq \ell^2(\mathbb{Z}^n)$. We say J is *measurable* if $x \mapsto \langle P_J(x)u, v\rangle$ is measurable on $[0,1)^n$ for every $u, v \in \ell^2(\mathbb{Z}^n)$.

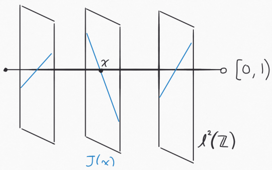


Example: $\mathbb{Z}^n \subseteq \mathbb{R}^n$

Given a measurable range function

$J\colon [0,1)^n \to \{\text{closed subspaces of } \ell^2(\mathbb{Z}^n)\},$

define



$$M_J = \{\varphi \in L^2([0,1)^n; \ell^2(\mathbb{Z}^n)) : \varphi(x) \in J(x) \text{ for a.e. } x \in [0,1)^n\}.$$

It's modulation invariant: for any $\varphi \in M_J$, we have

$$e^{-2\pi i k\cdot x} \cdot \varphi(x) \in J(x) \qquad \text{for a.e. } x \in [0,1)^n.$$

Helson & Srinivasan (1964): Those are all the MI spaces!

Example: $\mathbb{Z}^n \subseteq \mathbb{R}^n$

Fiberization operator $\mathcal{T}\colon L^2(\mathbb{R}^n) \to L^2([0,1)^n; \ell^2(\mathbb{Z}^n))$,

$$(\mathcal{T}f)(x) = \left\{\hat{f}(x+k)\right\}_{k\in\mathbb{Z}^n} \qquad \left(f \in L^2(\mathbb{R}^n),\ x \in [0,1)^n\right)$$

Corollary (de Boor, DeVore, and Ron, 1994)

SI subspaces of $L^2(\mathbb{R}^n)$ are indexed by measurable range functions

$$J\colon [0,1)^n \to \{\textit{closed subspaces of } \ell^2(\mathbb{Z}^n)\}.$$

The subspace associated with J is

$$V_J = \{f \in L^2(\mathbb{R}^n) : (\mathcal{T}f)(x) \in J(x) \textit{ for a.e. } x \in [0,1)^n\}.$$

Example: $\mathbb{Z}^n \subseteq \mathbb{R}^n$

Theorem (Bownik, 2000)

Fix $\mathscr{A} = \{f_j\}_{j\in I} \subseteq L^2(\mathbb{R}^n)$, *and let* $J\colon [0,1)^n \to \{$*closed subspaces of* $\ell^2(\mathbb{Z}^n)\}$ *be given by*

$$J(x) = \overline{\text{span}}\{(\mathcal{T} f_j)(x) : j \in I\} \subseteq \ell^2(\mathbb{Z}^n) \qquad (x \in [0,1)^n)\,.$$

Then $S(\mathscr{A}) = V_J$*. For constants* $A, B > 0$*, TFAE:*

1. *The integer shifts* $\{L_k f_j\}_{j\in I, k\in\mathbb{Z}^n}$ *form a frame for* V_J *w/ bounds* A, B*.*
2. *For a.e.* $x \in [0,1)^n$, $\{(\mathcal{T} f_j)(x)\}_{j\in I}$ *is a frame for* $J(x)$ *w/ bounds* A, B*.*

Example: $\mathbb{Z}^n \subseteq \mathbb{R}^n$, one generator

Corollary (Benedetto & Li, 1998)

Given $f \in L^2(\mathbb{R}^n)$, denote $S(f) = \overline{\operatorname{span}}\{f(\cdot + k) : k \in \mathbb{Z}^n\}$. Then the following are equivalent for any constants $A, B > 0$:

1. The integer shifts $\{f(\cdot + k) : k \in \mathbb{Z}^n\}$ produce a frame for $S(f)$ with bounds A, B.
2. For a.e. $x \in [0,1)^n$, $\displaystyle\sum_{k \in \mathbb{Z}^n} \left|\hat{f}(x+k)\right|^2 \in \{0\} \cup [A, B]$.

Proof.

The range function for $S(f)$ is $J(x) = \operatorname{span}\{(\mathcal{T}f)(x)\}$. Given $x \in [0,1)^n$, we have that $\{(\mathcal{T}f)(x)\}$ is a frame for $J(x)$ if and only if:

- $(\mathcal{T}f)(x) = 0$, hence $J(x) = \{0\}$, or
- $A \leq \|(\mathcal{T}f)(x)\|^2 \leq B$.

Now observe that $\|(\mathcal{T}f)(x)\|^2 = \sum_{k \in \mathbb{Z}^n} \left|\hat{f}(x+k)\right|^2$.

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Extensions to locally compact abelian (LCA) groups

What about general $H \subseteq G$, both LCA?

- Cabrelli and Paternostro (2010), Kamyabi Gol and Raisi Tousi (2010): Generalize to G second countable LCA, $H \subseteq G$ closed and discrete, and G/H is compact.
- Bownik and Ross (2013): Remove "H is discrete" above. Still need G abelian and G/H compact.
- Bad news: Can't do things like $\mathbb{R}^m \subseteq \mathbb{R}^n$ or $\mathbb{Z}^m \subseteq \mathbb{R}^n$ when $m < n$.

Cast of characters

Dual group $\hat{G}$: All cont's homomorphisms $\alpha\colon G \to \mathbb{T}$,

$$(\alpha + \beta)(x) := \alpha(x) \cdot \beta(x) \qquad (\alpha, \beta \in \hat{G};\, x \in G)$$

Fourier transform $\mathcal{F}_G\colon L^2(G) \to L^2(\hat{G})$,

$$(\mathcal{F}_G f)(\alpha) = \hat{f}(\alpha) = \int_G f(x)\overline{\alpha(x)}\, dx \qquad \left(\alpha \in \hat{G}\right)$$

Plancherel measure on $\hat{G}$: $\mathcal{F}_G$ is unitary

Annihilator $H^* = \left\{\alpha \in \hat{G} : \alpha(y) = 1 \text{ for all } y \in H\right\}$

What is the fiberization operator doing?

$\mathcal{T}\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{T}^n; \ell^2(\mathbb{Z}^n))$,

$$(\mathcal{T}f)(x) = \left\{\hat{f}(x+k)\right\}_{k\in\mathbb{Z}^n} \qquad \left(f \in L^2(\mathbb{R}^n),\, x \in [0,1)^n\right).$$

1. Apply the Fourier transform $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.
2. Do the periodization trick

$$\int_{\mathbb{R}^n} f(x)\, dx = \int_{[0,1)^n} \sum_{k\in\mathbb{Z}^n} f(x+k)\, dx$$

to identify $L^2(\mathbb{R}^n) \cong L^2([0,1)^n; \ell^2(\mathbb{Z}^n))$. Basically, think of $[0,1)^n$ as being $(\mathbb{R}^n)^\wedge/(\mathbb{Z}^n)^*$, unrolled in $(\mathbb{R}^n)^\wedge$.

Why do we like co-compactness?

Problem: When G/H is not compact, H^* is not discrete. Then we can't think of $\hat{G}/H^*$ as a set in $\hat{G}$ with positive measure.

Bottom line

When G/H is not compact, it's not clear how to define fiberization.

Solution: Don't use the $\hat{G}$-measure for $\hat{G}/H^*$!

Lemma (Feldman and Greenleaf, 1968)

There is a Borel measurable cross-section $\tau\colon \hat{G}/H^* \to \hat{G}$ whose image $\Omega := \tau(\hat{G}/H^*)$ intersects each coset of H^* exactly once.

Invariant measure on $\hat{G}/H^*$:

$$\int_{\hat{G}} f(\alpha)\, d\alpha = \int_{\hat{G}/H^*} \int_{H^*} f(\kappa + \alpha)\, d\alpha\, d(\kappa + H^*) \qquad \Big(f \in L^1(\hat{G})\Big).$$

Measure space isomorphism: $\hat{G}/H^* \times H^* \cong \hat{G}$,

$$(\alpha + H^*, \kappa) \mapsto \kappa + \tau(\alpha + H^*) \qquad (\alpha + H^* \in \hat{G}/H^*, \kappa \in H^*)$$

Hilbert space isomorphism: $L^2(\hat{G}) \cong L^2(\hat{G}/H^* \times H^*) \cong L^2(\hat{G}/H^*; L^2(H^*))$

Fiberization operator

Definition

The *fiberization* operator

$$\mathcal{T}\colon L^2(G) \to L^2(\hat{G}/H^*; L^2(H^*))$$

is given by

$$(\mathcal{T}f)(\alpha H^*)(\kappa) = \hat{f}(\kappa + \tau(\alpha + H^*)) \qquad (f \in L^2(G), \alpha + H^* \in \hat{G}/H^*, \kappa \in H^*).$$

Basically: Apply Fourier transform for G, then split off H^*.

Key properties: (1) unitary; (2) H-shifts $\to$ modulations

$$(\mathcal{T}L_y f)(\alpha + H^*) = \overline{\alpha(y)} \cdot (\mathcal{T}f)(\alpha)$$

Example: $\mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$

$$G = \mathbb{R}^2 \qquad \hat{G} \cong \mathbb{R}^2 \qquad \hat{G}/H^* \cong \mathbb{R}$$

$$H = \mathbb{R} \times \{0\} \qquad H^* \cong \{0\} \times \mathbb{R} \qquad \Omega \cong \mathbb{R} \times \{0\}$$

$$\mathcal{T} \colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R}; L^2(\mathbb{R})),$$

$$(\mathcal{T}f)(x) = \hat{f}(x, \cdot) \qquad \big(f \in L^2(\mathbb{R}^2),\, x \in \mathbb{R}\big)$$

Example: $\mathbb{Z} \times \{0\} \subseteq \mathbb{R}^2$

$$G = \mathbb{R}^2 \qquad \hat{G} \cong \mathbb{R}^2 \qquad \hat{G}/H^* \cong \mathbb{T}$$

$$H = \mathbb{Z} \times \{0\} \qquad H^* \cong \mathbb{Z} \times \mathbb{R} \qquad \Omega \cong [0,1) \times \{0\}$$

$$\mathcal{T} \colon L^2(\mathbb{R}^2) \to L^2([0,1); L^2(\mathbb{Z} \times \mathbb{R})),$$

$$(\mathcal{T}f)(x)(k,y) = \hat{f}(x+k,y) \qquad \big(f \in L^2(\mathbb{R}^2),\ x \in [0,1),\ k \in \mathbb{Z},\ y \in \mathbb{R}\big)$$

Classification of SI spaces

Fiberization operator $\mathcal{T}\colon L^2(G) \to L^2(\hat{G}/H^*;\, L^2(H^*))$

Theorem (JI)

H-SI spaces in $L^2(G)$ are indexed by measurable range functions

$$J\colon \hat{G}/H^* \to \{\textit{closed subspaces of } L^2(H^*)\}.$$

The subspace corresponding to J is

$$V_J = \{f \in L^2(G) : (\mathcal{T}f)(\alpha + H^*) \in J(\alpha + H^*) \textit{ for a.e. } \alpha + H^* \in \hat{G}/H^*\}.$$

Characterization of SI frames

Fiberization operator $\mathcal{T}\colon L^2(G) \to L^2(\hat{G}/H^*;\, L^2(H^*))$

Theorem (JI)

Fix $\mathscr{A} = \{f_j\}_{j\in I} \subseteq L^2(G)$*, and let*
$J\colon \hat{G}/H^* \to \{$*closed subspaces of* $L^2(H^*)\}$ *be given by*

$$J(\alpha{+}H^*) = \overline{\operatorname{span}}\{(\mathcal{T}f_j)(\alpha{+}H^*) : j \in I\} \subseteq L^2(H^*) \qquad \left(\alpha + H^* \in \hat{G}/H^*\right).$$

Then $S(\mathscr{A}) = V_J$*. For constants* $A, B > 0$*, TFAE:*

1. *The H-shifts* $\{L_y f_j\}_{j\in I, y\in H}$ *form a frame for* V_J *with bounds* A, B*.*
2. *For a.e.* $\alpha + H^* \in \hat{G}/H^*$, $\{(\mathcal{T}f_j)(\alpha + H^*)\}_{j\in I}$ *is a frame for* $J(\alpha + H^*)$ *with bounds* A, B*.*

Outline

An alternative to fiberization

The *Zak transform* is a unitary $Z\colon L^2(\mathbb{R}^n) \to L^2([0,1)^n \times [0,1)^n)$,

$$(Zf)(t,s) = \sum_{k\in\mathbb{Z}^n} f(t+k)e^{-2\pi i k\cdot s} \qquad (s,t \in [0,1)^n).$$

- First used separately by Gelfand (1950) and Weil (1964). Zak rediscovered it later.
- Key properties: (1) Unitary, (2) integer shifts $\to$ modulation

$$(ZL_k f)(s) = e^{-2\pi i k\cdot s}\cdot (Zf)(s) \qquad (f \in L^2(\mathbb{R}^n), k \in \mathbb{Z}^n, s \in [0,1)^n).$$

An alternative to fiberization

The *Zak transform* is a unitary $Z\colon L^2(\mathbb{R}^n) \to L^2([0,1)^n; L^2([0,1)^n))$,

$$(Zf)(s)(t) = \sum_{k \in \mathbb{Z}^n} f(t+k) e^{-2\pi i k \cdot s} \qquad (s,t \in [0,1)^n).$$

- First used separately by Gelfand (1950) and Weil (1964). Zak rediscovered it later.
- Key properties: (1) Unitary, (2) integer shifts $\to$ modulation

$$(ZL_k f)(s) = e^{-2\pi i k \cdot s} \cdot (Zf)(s) \qquad (f \in L^2(\mathbb{R}^n), k \in \mathbb{Z}^n, s \in [0,1)^n).$$

An alternative to fiberization

Fix a Borel cross-section $\tau\colon G/H \to G$, i.e. transversal $\Omega = \tau(G/H) \subseteq G$.

Definition

Given $f\colon G \to \mathbb{C}$ and $x + H \in G/H$, define $f_{x+H}\colon H \to \mathbb{C}$ by

$$f_{x+H}(y) = f(\tau(x) + y) \qquad (y \in H).$$

The *Zak transform* is the unitary $Z\colon L^2(G) \to L^2(\hat{H}; L^2(G/H))$ given by

$$(Zf)(\alpha)(x + H) = f_{x+H}\hat{\ }(\alpha) \qquad (f \in L^2(G),\ \alpha \in \hat{H},\ x + H \in G/H).$$

Basically: Treat cosets like copies of H, and apply the Fourier transform.

Key property: H-shifts $\to$ modulation

$$(ZL_y f)(\alpha) = \overline{\alpha}(y) \cdot (Zf)(\alpha)$$

Example: $\mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$

$$G = \mathbb{R}^2 \qquad G/H \cong \mathbb{R} \qquad \hat{H} \cong \mathbb{R}$$
$$H = \mathbb{R} \times \{0\} \qquad \Omega = \{0\} \times \mathbb{R}$$

$$(f_x)(t) = f(t,x) \qquad (f \in L^2(\mathbb{R}^2),\ x, t \in \mathbb{R})$$

$$Z\colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R}; L^2(\mathbb{R})),$$
$$(Zf)(\xi)(x) = (f_x)\hat{}\,(\xi) = \int_{\mathbb{R}} f(t,x) e^{-2\pi i t\xi}\, dt \qquad (x, \xi \in \mathbb{R})$$

Example: $\mathbb{Z} \times \{0\} \subseteq \mathbb{R}^2$

$$G = \mathbb{R}^2 \qquad G/H \cong \mathbb{T} \times \mathbb{R} \qquad \hat{H} \cong \mathbb{T} \cong [0,1)$$

$$H = \mathbb{Z} \times \{0\} \qquad \Omega = [0,1) \times \mathbb{R}$$

$$(f_{x,y})(k) = f(x+k,y) \qquad \big(f \in L^2(\mathbb{R}^2),\ x \in [0,1),\ y \in \mathbb{R},\ k \in \mathbb{Z}\big)$$

$$Z\colon L^2(\mathbb{R}^2) \to L^2([0,1); L^2([0,1) \times \mathbb{R})),$$

$$(Zf)(t)(x,y) = (f_{x,y})\hat{}\,(e^{2\pi ti}) = \sum_{k \in \mathbb{Z}} f(x+k,y)e^{2\pi kti} \quad (x,t \in [0,1);\ y \in \mathbb{R})$$

The Zak transform and SI spaces

Zak transform $Z\colon L^2(G) \to L^2(\hat{H}; L^2(G/H))$

Theorem (JI)

H-SI spaces in $L^2(G)$ are indexed by measurable range functions

$$J\colon \hat{H} \to \{\text{closed subspaces of } L^2(G/H)\}.$$

The space associated with J is

$$V_J = \{f \in L^2(G) : (Zf)(\alpha) \in J(\alpha) \text{ for a.e. } \alpha \in \hat{H}\}.$$

Zak transform analysis of H-SI frames

Theorem (JI)

Fix $\mathscr{A} = \{f_j\}_{j \in I} \subseteq L^2(G)$, *and let* $J\colon \hat{H} \to \{$*closed subspaces of* $L^2(G/H)\}$ *be given by*

$$J(\alpha) = \overline{\operatorname{span}}\{(Zf_j)(\alpha) : j \in I\} \subseteq L^2(G/H) \qquad \left(\alpha \in \hat{H}\right).$$

Then $S(\mathscr{A}) = V_J$. *For constants* $A, B > 0$, *TFAE:*

1. *The H-shifts* $\{L_y f_j\}_{j \in I, y \in H}$ *form a frame for* V_J *with bounds* A, B.
2. *For a.e.* $\alpha \in \hat{H}$, $\{(Zf_j)(\alpha)\}_{j \in I}$ *is a frame for* $J(\alpha)$ *with bounds* A, B.

Example: $\mathbb{Z}^n \subseteq \mathbb{R}^n$

Corollary (JI; cf. Benedetto & Li, 1998)

Let $Z\colon L^2(\mathbb{R}^n) \to L^2([0,1)^n \times [0,1)^n)$ be the Zak transform as usually defined,

$$(Zf)(t,s) = \sum_{k \in \mathbb{Z}^n} f(t+k) e^{-2\pi i k \cdot s} \qquad (s,t \in [0,1)^n).$$

Given $f \in L^2(\mathbb{R}^n)$, denote $S(f) = \overline{\operatorname{span}}\{f(\cdot + k) : k \in \mathbb{Z}^n\}$. Then the following are equivalent for any constants $A, B > 0$:

1. The integer shifts $\{f(\cdot + k) : k \in \mathbb{Z}^n\}$ produce a frame for $S(f)$ with bounds A, B.
2. For a.e. $s \in [0,1)^n$,

$$\int_{[0,1)^n} |Zf(t,s)|^2 \, dt \in \{0\} \cup [A,B].$$

Questions?