OPTIMUM SIGNAL SYNTHESIS FOR TIME-SCALE ESTIMATION

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ABSTRACT

In signal analysis, the joint estimation of the time-scale parameters which can affect a known signal (Doppler effect or scale effect, delay...) may be a problem of interest. An important result has shown that, even if the quality of the time delay estimation is classically given by the inverse spread of the signal spectral density, the quality of the scale estimation only depends on the inverse of the signal spread in Mellin space. This spread has a direct interpretation in the time-frequency plane and can be precisely estimated when duration, bandwidth and relative bandwidth of the signal are known. We propose here to develop two methods of optimum signal synthesis which minimize the variance of the estimates given by the Cramer-Rao lower bounds. The first method is based on the stationary phase principle, applied on frequency and Mellin spaces, which allows to construct signals with given auto-correlation functions in scale and time spaces. The second method is devoted to the construction of a frequency phase law depending on the mellin variable with the spreads in frequency and Mellin spaces related to the expected scale and time-delay resolutions.

1. FORMULATION OF THE PROBLEM

We are dealing with the problem of the joint estimation of the time-scale parameters of a known signal embedded in gaussian white noise. This is, for example, the case encountered in broad-band radar or sonar theory, when looking for parameters such as the velocity (related to scale parameter) or the position (related to time-shift parameter) of a target. In this case, the questions we are trying to answer are : which is the best signal to use for minimizing the variances of the estimates ? Can we develop synthesis methods which allow to construct such a signal ? The answer to the first question has already been developed in [6] and is briefly recalled here in order to develop the synthesis.

Let $z(t)$ be the transmitted and analytic signal. Its Fourier transform $Z(f)$ has therefore no negative frequency. The general transformation $x(t)$ of the signal $z(t)$ can be expressed as :

$$x(t, \theta_0) = A_0 T_{\theta_0} z(t)\, e^{i\phi_0} + b(t) \tag{1}$$

where T_{θ_0} is a time-scale action of the affine group which transforms the signal $z(t)$ with a set $\theta_0 = (a_0, b_0)$ of unknown parameters (time scale a_0 and time shift b_0). The parameter A_0 is the amplitude, ϕ_0 a phase change and $b(t)$ a zero-mean white gaussian noise with σ^2 variance. When the probability density of the parameters A_0 and ϕ_0 is unknown, the Maximum Likelihood ratio Λ to maximize, according to the Maximum Likelihood estimation theory, is given by the square modulus of the broad-band cross-ambiguity function :

$$\Lambda(\theta_0, \theta) = \frac{1}{2\sigma^2} \left| \int_{-\infty}^{+\infty} x(t, \theta_0)\, T_\theta^* z(t)\, dt \right|^2 \tag{2}$$

The efficiency of an estimate $\hat{\theta_0}$ is generally measured by its variance var$(\hat{\theta_0})$. For an unbiased estimate ($E(\hat{\theta_0}) = \theta_0$), this variance has a lower value given by the Cramer Rao Bounds (CRB) [8]. The CRB are obtained by inverting the Fisher Information Matrix (FIM) defined as :

$$J_{i,j} = \left(-E \left[\frac{\partial^2 \Lambda}{\partial \theta_i \partial \theta_j} \right] \right)_{i,j} \tag{3}$$

where θ_i denotes each component of the vector θ. The time-scaled and time shifted signal $x(t)$ can be put in the form :

$$x(t) = A_0\, z(a_0^{-1} t - b_0)\, e^{i\phi_0} + b(t) \tag{4}$$

The statistic to maximize is given by the square modulus of the broad-band cross-ambiguity function which is rewritten in the frequency domain :

$$\Lambda = \frac{a}{2\sigma^2} \left| \int_0^{+\infty} X(f)\, Z^*(af)\, e^{2i\pi abf}\, df \right|^2 \tag{5}$$

where the parameters a and b represent the scale factor and the time shift parameters to estimate. In the next section and using the Mellin transform [2, 5], the FIM computation is easily performed and leads to a perfect physical interpretation of its coefficients in the time-frequency half plane.

2. TIME SCALE ESTIMATION

2.1. The Mellin Transform

The Mellin transform which plays an important part in the computation and the physical interpretation of the FIM's coefficients have been well defined in [2] and acts on the analytic signal $Z(f)$ in frequency by :

$$M^\xi[Z](\beta) = \int_0^{+\infty} Z(f)\, e^{2i\pi\xi f}\, f^{2i\pi\beta + r}\, df \tag{6}$$

This transform can be interpreted as the coefficient of the decomposition of the signal onto a hyperbolic signals basis with a group delay law given in the time-frequency half plane by the equation $t = \xi + \beta/f$ with the invariant scalar product given by :

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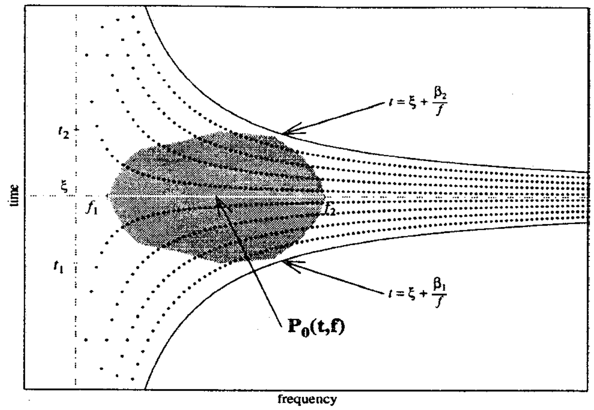

Figure 1: Localization in the time-frequency half plane of a signal $Z(f)$ having a time-frequency energy distribution $P_0(t, f)$. The two hyperbolas defined by equations $t = \xi + \beta_1/f$ and $t = \xi + \beta_2/f$ delimit the support $[\beta_1, \beta_2]$ of its Mellin transform.

$$\int_0^{+\infty} Z_1(f)Z_2^*(f)f^{2r+1}df = \int_{-\infty}^{+\infty} M^\xi[Z_1](\beta)M^{\xi*}[Z_2](\beta)d\beta \tag{7}$$

The dual Mellin variable β therefore characterizes the coefficient of an hyperbola in the time-frequency half plane. The parameter r is free but is chosen here equal to $-1/2$ to preserve the classical scalar product. The study of the tomographic construction of the unitary affine time-frequency distribution $P_0(t, f)$ [1] has shown that a signal localized in the time-frequency half plane has a Mellin transform support bounded in Mellin space (cf. figure 1). The connection between the P_0 distribution :

$$\begin{aligned} P_0(t,f) &= f\int_{-\infty}^{+\infty}(\lambda(u)\lambda(-u))^{1/2}\, Z(\lambda(u)f) \\ &\quad Z^*(\lambda(-u)f)\, e^{-2i\pi ftu}\, du \end{aligned} \tag{8}$$

where $\lambda(u) = u\exp(-u/2)/2\sinh(u/2)$ and the Mellin transform is nothing but a hyperbolic Radon transform :

$$\int_{-\infty}^{+\infty} dt \int_0^{+\infty} P_0(t,f)\,\delta(t-\xi-\beta/f)\,\frac{df}{f} = \left|M^\xi[Z](\beta)\right|^2 \tag{9}$$

Using an a priori knowledge of the localization of the signal in the time-frequency half plane (bandwidth, relative bandwidth, duration), it is now possible to perfectly determine the spread $\sigma_\beta = \beta_2 - \beta_1$ of the signal in the Mellin space (cf. figure 1). In the following, the ξ parameter will be chosen equal to zero and the transform will be noted $M[Z](\beta)$. The main property of the Mellin transform is the property of scale invariance :

$$\begin{array}{ccl} Z(f) & \longrightarrow & Z'(f) = \sqrt{a}\,Z(af) \\ \downarrow & & \downarrow \\ M[Z](\beta) & \longrightarrow & M[Z'](\beta) = a^{-2i\pi\beta}M[Z](\beta) \end{array} \tag{10}$$

which is useful when rewriting (5) :

$$\Lambda = \frac{1}{2\sigma^2}\left|\int_{-\infty}^{+\infty} M[X](\beta)\, M^*[Z_b](\beta)\, a^{2i\pi\beta}\, d\beta\right|^2 \tag{11}$$

with $Z_b(f) = Z(f)\exp(2i\pi bf)$. Another important property of the Mellin transform, useful for computation of the FIM coefficients, is the diagonalization of the operator $\mathcal{B}$ defined by :

$$\mathcal{B}Z(f) = -\frac{1}{2i\pi}\left(f\frac{d}{df} + \frac{1}{2}\right) Z(f) \tag{12}$$

which is transformed as $M[\mathcal{B}Z](\beta) = \beta M[Z](\beta)$.

2.2. The Fisher Information Matrix

The Fisher Information Matrix has the following form [6] :

$$J = \frac{4\pi^2 A_0^2}{\sigma^2}\begin{pmatrix} \sigma_\beta^2 & f_0\beta_0 - M \\ f_0\beta_0 - M & \sigma_f^2 \end{pmatrix} \tag{13}$$

where the parameters σ_f, f_0 and A_0^2/σ^2 define respectively the spectrum bandwidth, the mean frequency of the signal, the ratio is the Signal-to-Noise Ratio and where the parameters β_0, σ_β are given by :

$$\beta_0 = \int_{-\infty}^{+\infty} \beta\, |M[Z](\beta)|^2\, d\beta \tag{14}$$

$$\sigma_\beta^2 = \int_{-\infty}^{+\infty} (\beta-\beta_0)^2\, |M[Z](\beta)|^2\, d\beta \tag{15}$$

The first and second order moments can be viewed respectively as the mean β and the spread of the signal Z in Mellin space. Cohen has called them respectively the mean scale and the scale bandwidth [3]. Suppose the spectrum be in the form $Z(f) = A(f)\exp(i\phi(f))$. Using the $\mathcal{B}$ operator defined in (12), the two quantities (14) and (15) can be transformed in the frequency domain :

$$\beta_0 = -\frac{1}{2\pi}\int_0^{+\infty} f\,\phi'(f)\,A^2(f)\,df \tag{16}$$

$$\sigma_\beta^2 = \frac{1}{4\pi^2}\left[\int_0^{+\infty}\left(f\,\frac{A'(f)}{A(f)} + \frac{1}{2}\right)^2 A^2(f)\,df \right. \tag{17}$$

$$\left. + \int_0^{+\infty} f^2\,\phi'^2(f)\,A^2(f)\,df\right] - \beta_0^2 \tag{18}$$

The two previous equations show that, in the frequency or time domains, the physical interpretation of such quantities is very difficult and this is for example the main problem encountered in an interesting paper [4] which deals with a similar subject. The geometrical help of the Mellin transform allows to easily understand the quantities 14 and 15.

By analogy with the narrow-band modulation index, the parameter M defined in (13) is called the broad-band modulation index and represents the rate of hyperbolic modulation. It has the expression :

$$M = -\frac{1}{2\pi}Im\int_0^{+\infty} f^2\,\frac{dZ}{df}\,Z^*(f)\,df \tag{19}$$

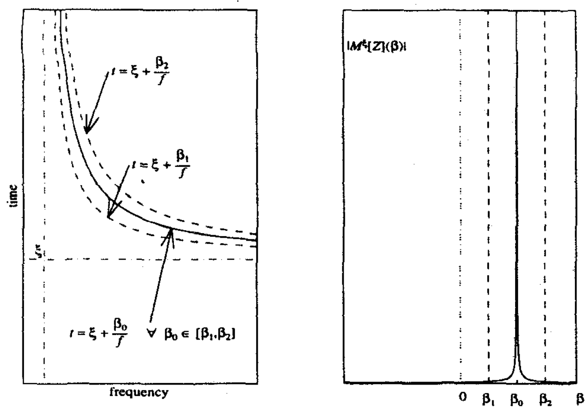

Figure 2: Localization in the time-frequency half plane of a hyperbolic signal $Z(f)$ labeled by its parameter β_0 and its Mellin transform. Any pair of hyperbolas with equation $t=\xi+\beta_1/f$ and $t=\xi+\beta_2/f$ (with $\beta_1<\beta_0<\beta_2$) can delimit the signal. Such a signal, although it has a infinite duration, has a zero-spread in Mellin space and therefore no scale resolution.

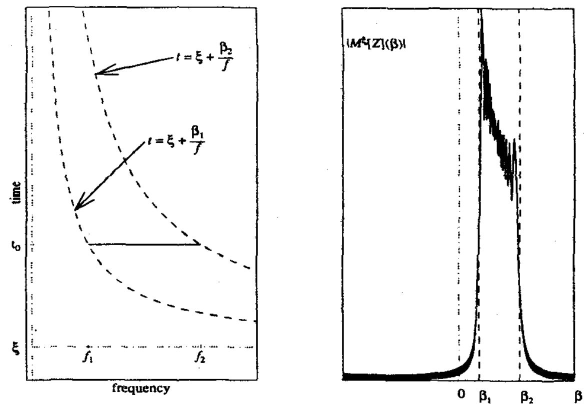

Figure 3: Localization in the time-frequency half plane of a short signal centered around $t=t_0$ with a bandwidth $B=f_2-f_1$ around $f_0=(f_1+f_2)/2$ and its Mellin transform. Such a signal, although of very short duration, has a spread $\sigma_\beta=(f_2-f_1)t_0$ in Mellin space and therefore a finite scale resolution.

To estimate the quality of the compression and delay parameters, the FIM must be inverted. Each term of the inverse matrix J^{-1} gives the variance lower bound of each estimate. As the estimates are unbiased and efficient (high SNR), the CRB are reached and the variance of the time delay estimate $\hat{b}$ and the scale estimate $\hat{a}$ are defined by :

$$\mathrm{var}(\hat{b})=\frac{\sigma^2}{4\pi^2A_0^2}\frac{\sigma_\beta^2}{\sigma_f^2\sigma_\beta^2-(M-\beta_0f_0)^2}\geq\frac{\sigma^2}{4\pi^2A_0^2}\frac{1}{\sigma_f^2} \quad (20)$$

$$\mathrm{var}(\hat{a})=\frac{\sigma^2}{4\pi^2A_0^2}\frac{\sigma_f^2}{\sigma_f^2\sigma_\beta^2-(M-\beta_0f_0)^2}\geq\frac{\sigma^2}{4\pi^2A_0^2}\frac{1}{\sigma_\beta^2} \quad (21)$$

The first result (20) shows that the time delay resolution is always related to the inverse of the signal spread in frequency. The result (21) is very important because it proves that the scale resolution depends only on the inverse of the signal spread in the Mellin space. As an example, let us consider the so-called Doppler invariant signals as hyperbolic signals (cf. figure 2) which are characterized by a no spread in Mellin space ($\sigma_\beta=0$) : this kind of signals does not lead to a good scale resolution. The figure 3 shows, unlike the previous figure, there is no contradiction between very short time duration and high scale resolution.

3. OPTIMAL SIGNALS SYNTHESIS

The first method minimizing the Cramer-Rao lower bounds is devoted to the construction of optimal signals with given autocorrelation functions in scale and delay spaces with control of the sidelobes. The second one determines a phase law which allows the signal to reach the desired spreads and resolutions in the Mellin and frequency spaces.

3.1. The Stationary Phase Method

The Stationary Phase Principle method already used for designing high time bandwidth product signals [7] is applied here but is extended to the Mellin and frequency spaces. The main idea is to construct high $\sigma_f\sigma_\beta$ product signals (asymptotic signals) in the same way. The inverse Mellin transform is defined by :

$$Z(f)=e^{-2i\pi\xi f}f^{-1/2}\int_{-\infty}^{+\infty}M^\xi[Z](\beta)\,f^{-2i\pi\beta}\,d\beta \quad (22)$$

Following the stationary phase principle method and applying it on (22), we have up to a constant phase :

$$Z(f)=\frac{e^{-2i\pi\xi f}}{\sqrt{f}}\sqrt{\frac{2\pi}{|\phi''(\lambda)|}}\left|M^\xi[Z](\lambda)\right|e^{i(\phi(\lambda)-2\pi\lambda\log f)} \quad (23)$$

where we note $M^\xi[Z](\beta)=\left|M^\xi[Z](\beta)\right|\exp(i\phi(\beta))$ and where λ is the stationary point defined by the following equation :

$$\frac{d}{d\beta}\left[\phi(\beta)-2\pi\beta\log f\right]_{\beta=\lambda}=0 \quad (24)$$

We found that the set of points $(\beta,F(\beta))$ for which the phase is stationary is defined by a relation which will be independently fixed in the second method of construction :

$$F(\beta)=\exp\left(\frac{1}{2\pi}\frac{d\phi(\beta)}{d\beta}\right) \quad (25)$$

If we note ϕ'^{-1} the reciprocal function of ϕ', the stationary point λ verifying (24) is defined by $\lambda=\phi'^{-1}(2\pi\log f)$. The spectrum phase law has therefore the form $\Psi(f)=-2\pi\xi f+\phi(\lambda)-2\pi\lambda\log f$ and is thus defined by its group delay :

$$T(f)=\xi+\frac{\lambda}{f} \quad (26)$$

Acting on the shape of $|Z(f)|$ and $|M^\xi[Z](\beta)|$ by judiciously choosing the time-shift autocorrelation function $R(b)$ and scale autocorrelation function $S(a)$ defined according to :

$$|Z(f)|^2=\int_{-\infty}^{+\infty}R(b)\,e^{-2i\pi bf}\,db \quad (27)$$

$$\left|M^{\xi}[Z](\beta)\right|^2 = \int_0^{+\infty} S(a)\, a^{-2i\pi\beta-1}\, da \tag{28}$$

we thus define the phase law $\phi(\lambda)$ given by the differential equation :

$$\phi''(\lambda) = 2\pi \frac{\left|M^{\xi}[Z](\lambda)\right|^2}{f\,|Z(f)|^2} \tag{29}$$

Choosing $\psi(\lambda) = \frac{1}{2\pi}\phi'(\lambda) = \log f$, the last equation can be integrated with respect to λ and leads to :

$$\int_0^{\exp\psi(\lambda)} |Z(f)|^2\, df = \int_{-\infty}^{\lambda} \left|M^{\xi}[Z](\beta)\right|^2 d\beta \tag{30}$$

By fixing a given λ, it is now possible by (30) to find $\psi(\lambda)$ and to determine the phase law $\phi(\beta)$ by :

$$\phi(\beta) = 2\pi \int_{-\infty}^{\beta} \psi(u)\, du \tag{31}$$

As an example, let us consider the problem of signal synthesis having an unit amplitude over a chosen bounds in frequency and Mellin spaces (the autocorrelation functions have thus both a $\sin x/x$ shape). The differential equation (29) to solve takes the following form :

$$\phi''(\lambda) = \frac{2\pi}{f} = 2\pi \exp\left(-\frac{\phi'(\lambda)}{2\pi}\right) \tag{32}$$

which, when A and B are constants fixed by initial conditions, gives the Mellin phase :

$$\phi(\beta) = 2\pi\left[(\beta + A)\log(\beta + A) - (A + \beta)\right] + B \tag{33}$$

3.2. Phase Law Construction

Consider a monochromatic and analytic signal given by its equation $Z(f) = \delta(f - f_0)$. This signal has a Mellin transform given by $M^{\xi}[Z](\beta) = f_0^{2i\pi\beta - 1/2}\, e^{2i\pi\xi f_0}$. We can therefore perfectly determine the frequency law of the signal $Z(f)$ as the function of the Mellin variable :

$$f_0 = \exp\left(\frac{1}{2\pi}\frac{d\phi}{d\beta}\right) \tag{34}$$

where $\phi(\beta)$ is the phase of the Mellin transform of Z. Extending this relation, we obtain, independently of the first method, the expression of the frequency in terms of the β variable :

$$F(\beta) = \exp\left(\frac{1}{2\pi}\frac{d\phi(\beta)}{d\beta}\right) \tag{35}$$

Given a frequency law $F(\beta)$ in Mellin space, we can obtain by solving (35) the derivative of the Mellin phase and finally the expression of the signal in Mellin space $M^{\xi}[Z](\beta) = e^{i\phi(\beta)}$. As an example, consider a linear phase law $F(\beta) = A\beta + B$ where A and B are parameters which control the frequency spread and the Mellin spread. Solving the differential equation (35), we found the Mellin phase already given by (33) :

$$\phi(\beta) = \frac{2\pi}{A}\left[(A\beta + B)\log(A\beta + B) - (A\beta + B)\right] + C \tag{36}$$

Using the inverse Mellin transform of the signal $\exp(i\phi(\beta))$, we can obtain the spectrum $Z(f)$. If the signal is supposed asymptotic (high $\sigma_f\sigma_\beta$ product), we can apply the previous result given by the stationary phase method and deduce by (26) the hyperbolic group delay of the spectrum $Z(f)$:

$$T(f) = \xi + \frac{F^{-1}(f)}{f} = \xi + \frac{f - B}{Af} \tag{37}$$

This procedure is the analogous construction of a signal from time to frequency space using the definition of the instantaneous frequency. It only ensures that the signal will have, at one and the same time, a given bandwidth σ_f and a given spread σ_β in Mellin space but does not ensure, unlike the first method, the sidelobes quality of the two autocorrelation functions in range and velocity spaces.

4. CONCLUSION

The analytical expression of the Cramer Rao bounds for the joint time-scale estimation has been established using the Mellin transform. An important result concerns the scale resolution which is related to the inverse of the spread of the signal in Mellin space. This spread has a direct geometrical interpretation in the time-frequency half plane and can be easily estimated when duration, bandwidth and relative bandwidth are known. Thanks to this interpretation, two interesting procedures have been proposed to construct optimal broad-band signals which minimize the Cramer-Rao lower bounds. These methods can be of interest for example for designing powerful radar or sonar broad-band waveforms.

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